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Keskusteluaiheita
Discussion papers

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COMPENSATED INCOME

IN CONSUMPTION-SAVING ECONOMIES

No. 66 6.11.1980

PREFACE

I consider in the paper how the purchasing power of income may be defined for an income user who saves a part of his income. The traditional definition of the purchasing power of "income" (i.e., consumption expenditure) or in other words the definition of the real income (consumption) as given in the consumer theory is no longer applicable as such in a consumption-saving economy. Of course, the real income in the consumption-saving economy is formally a generalization of the real income in all-consumption economies.

The paper is closely related to a joint work "Increase in sales taxes and the purchasing power of income" (in Finnish) by Pentti Vartia and the author, which will be published in the near future by ETLA. Actually it was planned to include the present paper as an appendix to our joint work, but now it seems more appropriate to publish these two works separately. Results presented in the present paper provide for instance a microeconomic justification for our analysis concerning the effects of an increase of sales taxes on the purchasing power of disposable income.

The present paper makes it possible to generalize the traditional "consumer surplus" techniques to an "income user surplus" analysis, which has numerous potential applications in the field of cost-benefit analysis.

I thank for useful comments given by fellow researchers Kari Alho, Jukka Lassila and Pentti Vartia.

Helsinki 14.11.1980

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COMPENSATED INCOME IN CONSUMPTION-SAVING ECONOMY

by Yrjö O. Vartia

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1. INTRODUCTION AND THEORETICAL PRELIMINARIES

In all-consumption economies all income is used in consumption and therefore saving is identically zero. The allocation of income Y (which equals here total consumption expenditure C) into different commodity categories is carried out according to the ordinary theory of consumer's choise: i.e. by maximizing a continuous, increasing and quasiconcave utility function $u(q) = u(q_1,\dots,q_n)$ under the budget constraint $\sum_{i=1}^{n} p_i q_i = p \cdot q \leq Y$, where $p = (p_1,\dots,p_n)$ is the vector of exogenous prices. By the increasingness (or nonsatiety) assumption the optimum $\hat{q}$ is attained in the boundary of budget constraint and therefore all income will be spent in consumption: $p \cdot \hat{q} = Y$. The demand functions are given by the demand system $\hat{q} = h(p,Y)$, where $h : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^{n}_{++}$ is the vector valued function of exogenous prices and income (or expenditure). The minimum cost of achieving a given utility level u when prices are p

$$(1) \qquad C(p,u) := \min\{\, C \mid C \geq p \cdot \bar{q} \ \&\ u(\bar{q}) = u \,\}$$

defines the cost function $C : \mathbb{R}^{n+1}_{++} \to \mathbb{R}_{++}$ in terms of prices and utility[1]. (Shortly: C(p,u) = "cost of utility" = cost of utility level u under p-prices). Sometimes the cost function is defined as a function C*(p,q) of prices p and quantities q determining the utility level:

1) The symbol: = means that the expression on left of it (here C(p,u)) is by definition equal to the expression.

$$(2) \qquad C^*(p,q) := C(p,u(q))$$

$$:= \min\{\, C \mid C \geq p \cdot \bar{q} \ \&\ u(\bar{q}) = u(q) \,\}.$$

The Konüs cost of living index[1]

$$(3) \qquad P(p^1,p^0;\ u) = \frac{C(p^1,u)}{C(p^0,u)}$$

tells how much more income is needed to buy a given utility level u in the case of p^1-prices than under p^0-prices. (Shortly: $P(p^1,p^0;\ u)$ = relative cost of utility u in two price situations). Usually the relative cost of the "old" utility level $u^0 = u(q^0)$, $q^0 = h(p^0,C^0)$ is calculated, which gives the Laspeyres' type Konüs cost of living index:

$$(4) \qquad P(p^1,p^0;\ u^0) = \frac{C(p^1,u^0)}{C(p^0,u^0)} = \frac{C(p^1,u^0)}{C^0} .$$

The Allen quantity index[2]

$$(5) \qquad Q(q^1,q^0;\ p) = \frac{C(p,u(q^1))}{C(p,u(q^0))} = \frac{C^*(p,q^1)}{C^*(p,q^0)}$$

1) See Diewert (1979, p. 5). Samuelson & Swamy (1974) call it the Economic Price Index.

2) In Diewert's (1979, p. 16) terminology. Samuelson & Swamy (1974) call it the Economic Quantity Index.

tells how much more income is needed the buy the utility level $u(q^1)$ determined by q^1 than that of q^0 under p-prices. (Shortly: $Q(q^1,q^0;\ p)$ = relative cost of two utility levels under p-prices). The Paasche's type Allen quantity index

$$(6)\qquad Q(q^1,q^0;\ p^1) = \frac{C(p^1,u(q^1))}{C(p^1,u(q^0))} = \frac{C^1}{C(p^1,u^0)}$$

and the Laspeyres' type of Konüs cost of living index (4) satisfy

$$(7)\qquad P(p^1,p^0;\ u^0)Q(q^1,q^0;\ p^1) = \frac{C^1}{C^0}.$$

Therefore (6) results by deflating the expenditure ratio C^1/C^0 by (4). This is the familiar procedure to calculate the quantity index by deflating the value ratio by the price index. It shows that if the value ratio C^1/C^0 is known and the price index $P(p^1,p^0;\ u^0)$ may be approximated accurately (as is usually the case) then also the quantity index $Q(q^1,q^0;\ p^1)$ is well approximated. Especially if $P(p^1,p^0;\ u^0) = C^1/C^0$ then $Q(q^1,q^0;\ p^1) = 1$, which implies that q^1 and q^0 lie on the some utility surface, and are indifferent. This is the situation when the change $C^0 \rightarrow C^1$ in expenditure is all needed to compensate for the price change $p^0 \rightarrow p^1$.

In addition the quantity index $Q(q^1,q^0;\ p^1) = [C^1/C^0]/P(p^1,p^0;\ u^0)$ exceeds (falls short of) 1 if and only if q^1 lies on a higher (lower) utility surface than q^0, see Vartia (1978b) and Diewert (1979, p. 16).

This far we have considered only all-consumption economies. We have given a general method to determine whether a given $q^1 = h(p^1,C^1)$ is preferred to $q^0 = h(p^0,C^0)$ or not: if the actual relative change in expenditure $(\frac{C^1}{C^0} - 1)$ exceeds the relative change in prices, $P(p^1,p^0;\ u^0) - 1$. The Konüs cost of living index $P(p^1,p^0;\ u^0)$ makes it possible to calculate also a hypothetical income (or expenditure)

$$(8)\qquad \bar{C}^1 = P(p^1,p^0;\ u^0)C^0\ ,$$

which is just enough for the consumer to maintain the old level of consumption although the prices have changed, $p^0 \rightarrow p^1$. Here $\bar{C}^1$ is the <u>compensated expenditure</u> (compensating for the price change), $\bar{C}^1 - C^0$ is the <u>compensation</u> in money units (say dollars) and $100(\frac{\bar{C}^1}{C^0} - 1) = 100(P(p^1,p^0;\ u^0) - 1)$ is the compensation in per cents. E.g. if $P(p^1,p^0;\ u^0) = 1.10$ then 10 % more income is needed to buy the old level of consumption.

In an alternative approach the <u>indirect utility function</u>

$$(9)\qquad v(p,C) = \max\{\,u(q) \mid p\cdot q \leq C\,\} = \max_{p\cdot q\leq C} u(q) = u(h(p,C))$$

is applied. Indirect utility function $(p,C) \rightarrow v(p,C)$ gives the maximum utility $u(q)$ attainable with p-prices and consumption expenditure C. If a direct utility function $q \to u(q)$ is given, then $v(p,C)$ may be calculated by (9). Alternatively, if a function $v: \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ with characteristic properties of an indirect utility function to be listed later is given, then the direct utility function $u: \mathbb{R}^n_{++} \to \mathbb{R}$ may be defined as follows ($r = p/C$ = vector of normalized prices)

$$(10) \qquad u(q) = \min_p \{ v \mid v = v(p,C) \;\&\; p \cdot q \le C \}$$
$$= \min_r \{ v(r,1) \mid r \cdot q \le 1 \}$$
$$= \min_{r \cdot q \le 1} v(r,1) .$$

Or define the (normalized shadow) price function $q \to \psi(q)$ as the vector of normalized prices $\hat{r} = \psi(q)$ which minimizes $v(r,1)$ under condition $r \cdot q \le 1$ for a given q. Then we have

$$(11) \qquad u(q) = v(\psi(q),\, 1) ,$$

which is dual to $v(p,C) = u(h(p,C))$ or $v(r,1) = u(h(r,1))$.

In differentiable cases the demand function $\hat{q} = h(r,1) = h(p/C,1)$ and the price function (or the inverse demand function) $\hat{r} = \psi(q)$ are determined straightforwardly by <u>Roy's Theorem</u>

$$(12) \qquad h(r,1) = \frac{\nabla v(r,1)}{\nabla v(r,1) \cdot r} = \frac{(v_1(r,1),\ldots,v_n(r,1))}{\sum_{i=1}^{n} v_i(r,1) r_i} ,$$

where $v_i(r,1) = \partial v(r,1)/\partial r_i$ and by <u>Wold's Theorem</u>

$$(13) \qquad \psi(q) = \frac{\nabla u(q)}{\nabla u(q) \cdot q} = \frac{(u_1(q),\ldots,u_n(q))}{\sum_{i=1}^{n} u_i(q) q_i} .$$

Note that the (<u>direct</u> or ordinary) demand function $\hat{q} = h(r,1)$ is calculated by Roy's Theorem from the <u>indirect</u> utility function $v(r,1)$, while the <u>inverse</u> demand function $\hat{r} = \psi(q)$ is calculated by Wold's Theorem from the (<u>direct</u>) utility function $u(q)$.

Therefore to derive the direct demand function $h(r,1) = h(p,C)$, where $p = Cr$, the knowledge of an indirect utility function $v(r,1)$ is most convenient. Next we list some <u>sufficient</u> properties which quarantee a function $r \to v(r,1)$ from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ to be an indirect utility function, see Afriat (1972, p. 34), Diewert (1974, 1979), Blackorby, Primont and Russel (1978) or Weymark (1979). Note that $v(r,1) = v(p,C)$, where $p = Cr$, gives the definition of $v(p,C)$ for any $(p,C) \in \mathbb{R}^n_{++}$. We list the relevant properties both for $v(r,1)$ and $v(p,C)$.

<u>Conditions A on v(r,1):</u>

A1: $v(r,1)$ is a continuous function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$

A2: $v(r,1)$ is decreasing, i.e. if $\bar{r}_i > r_i$ for all i, then $v(\bar{r},1) < v(r,1)$.

A3: $v(r,1)$ is quasiconvex, i.e. the set of normalized prices r worth of v at most $W(v) = \{ r \mid v(r,1) \le v\}$ is convex for any $v \in \mathbb{R}$ (W comes from worse).

The level sets or indifference surfaces of $v(r,1)$ look the same as those of the direct utility function $u(q)$, i.e. they are totally "above" any of their tangent planes.

Conditions $\bar{A}$ on $v(p,C)$:

$\bar{A}1$: $v(r,1)$ satisfies Conditions A

$\bar{A}2$: $v(p,C)$ is homogenous of degree zero, i.e. $v(\lambda p,\lambda C) = v(p,C)$ for all $\lambda > 0$.

From Conditions $\bar{A}$ it follows e.g. that $v(p,C)$ is decreasing in prices p, increasing in consumption expenditure C and quasiconvex with respect to prices p for any C.

Omitting some technical problems connected with the boundary of $\mathbb{R}^n_{++}$ (i.e. zeros and infinities, see Diewert (1974, 1979)) any indirect utility $v(r,1)$ satisfying A determines by use of (10) a unique direct utility function $u(q)$. The resulting function $u(q)$ satisfies

Conditions B on $u(q)$:

B1: $u(q)$ is a continuous function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$

B2: $u(q)$ is increasing: i.e. if $\bar{q}_i > q_i$ for all i, then $u(\bar{q}) > u(q)$.

B3: $u(q)$ is quasiconcave, i.e. the set of quantities q leading to utility u at least $B(u) = \{q|u(q) \geq u\}$ is convex for any $u \in \mathbb{R}$ (B comes from better).

Respectively, for any utility function $u(q)$ satisfying Conditions B a indirect utility function $v(p,C)$ (or $v(r,1)$) satisfying Conditions A' (or A) may be derived from (9). Therefore direct and indirect utility functions are equally appropriate means of representing consumer's preferences. This is the essence of "duality" in the theory of consumer's choice.

For purposes of later reference we call a function $h(p,C)$ from a region $\Omega^* \subset \mathbb{R}^{n+1}_{++}$ into $\mathbb{R}^n_+$ a demand function if it satisfies

BC. Budget condition: $\forall (p,C) \in \Omega^* : p \cdot h(p,C) \leq C$.

A demand function may or may not satisfy also

B. Balance: $\forall (p,C) \in \Omega^* : p \cdot h(p,C) = C$,

H. Homogeneity of degree zero: $\forall (p,C) \in \Omega^* : \forall \lambda > 0 :$ $h(\lambda p,\lambda C) = h(p,C) = h(p/C,1)$,

see Vartia (1978) and its references. A demand function $h(p,C)$ may or may not be associated with some utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$. We say that a utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ represents a demand function $h : \Omega^* \to \mathbb{R}^n_+$ if $h(p,C)$ is the unique u-maximal element in any budget set $B(p,C) = \{q|p \cdot q \leq C\}$, i.e. for all $(p,C) \in \Omega^*$: $\forall q \in B(p,C) : q \neq h(p,C) \Rightarrow u(q) < u(h(p,C))$.

A demand function $h(p,C)$ is representable by a utility function if there exists some utility function $u(q)$ representing it. For our present purposes it is sufficient to consider only utility functions $u(q)$ satisfying conditions B. This will give rise to the following (rather strong) utility hypothesis:

UH: Utility hypothesis

The demand function $h : \Omega^* \to \mathbb{R}^n_+$ is representable by a utility function $u(q)$ satisfying conditions B.

Because an indirect utility function $v(p,C)$ representing the same preferencies exists by duality a demand function $h(p,C)$ satisfying UH has in the differentiable case also a representation $h(p,C) = h(r,1) = \nabla v(r,1)/\nabla v(r,1)\cdot r = C\nabla v(p,C)/p.\nabla v(p,C$. It satisfies necessarily also B and H.

We return to the compensated expenditure in the case of price change $p^0 \to p^1$. In the base situation prices p^0 and expenditure C^0 determine the consumption bundle $q^0 = h(p^0,C^0)$. The old utility level is $u^0 = u(q) = u(h(p^0,C^0)) = v(p^0,C^0) = v^0$. The compensated income $\bar{C}^1$ given in (8) may be determined equally well as the solution of

$$(14) \qquad v(p^0,C^0) = v(p^1,\bar{C}^1) .$$

Equation (14) says, that under new prices p^1 expenditure $\bar{C}^1$ is just enough to maintain the old utility level. Because $v(p^1,C)$ is increasing in C any $C > \bar{C}^1$ would result to higher utility. An alternative but equivalent definition to (4) would therefore be

$$(15) \qquad P(p^1,p^0;\ u^0) = \frac{\bar{C}^1}{C^0},$$

where $\bar{C}^1$ satisfies (14).

Usually the old consumption bundle q^0 costs more than $\bar{C}^1$ dollars under p^1-prices, because by changing the old consumption pattern in the face of new relative prices the consumer may adjust himself to the new situation. Or in other words: the Laspeyres price index

$$(16) \qquad P_L(p^1,p^0,q^1,q^0) = \frac{p^1\cdot q^0}{p^0\cdot q^0} = \sum_{i=1}^{n} w_i^0(p_i^1/p_i^0) ,$$

where $w_i^0 = p_i^0q_i^0/p^0\cdot q^0 = v_i^0/C^0$ = the old value share, always exceeds the 'true' cost of living, i.e.

$$(17) \qquad P_L(p^1,p^0,q^1,q^0) \geq P(p^1,p^0;\ u^0) .$$

Therefore

$$(18) \qquad \hat{C}^1 = P_L(p^1,p^0,q^1,q^0)C^0$$

$$= [\Sigma\, w_i^0(p_i^1/p_i^0)]C^0$$

is more than is needed to retain the old level of consumption, i.e. shortly $\hat{C}^1 \geq \bar{C}^1$.

2. COMPLICATIONS CAUSED BY SAVING

But if saving S appear, so that the income $Y = C + S$ no longer equals the total consumption expenditure C, then we cannot apply without additional comments previous results. Although it is a common practise to deflate the income ratio Y^1/Y^0 by some price index P_0^1 (usually by the official consumer price index calculated by Laspeyres' formula) in order to 'calculate' (or rather to 'define') the change in real income, this (essentially sound) practise does not straightly follow from the theory of consumer's choice. Their connections are, however, quite apparent. Let's denote by $\tilde{q}^0$ ($\tilde{q}^1$) the consumption bundle which would be bought by total income $Y^0 = C^0 + S^0$ ($Y^1 = C^1 + S^1$) under p^0-prices (p^1-prices): i.e. $\tilde{q}^0 = h(p^0, Y^0)$ and $\tilde{q}^1 = h(p^1, Y^1)$. These bundles are usually different to the actual bundles $q^0 = h(p^0, C^0)$ and $q^1 = h(p^1, C^1)$ bought by consumption expenditures C^0 and C^1 in the same situations. For instance $\tilde{q}^0 = h(p^0, Y^0) = h(p^0, C^0 + S^0) \succ h(p^0, C^0) = q^0$ if $Y^0 > C^0$ or $S^0 = Y^0 - C^0 > 0$. In the case of negative saving $Y^0 < C^0$ and $h(p^0, Y^0) \prec h(p^0, C^0)$. The consumption bundles

$$(19) \qquad \tilde{q}^0 = h(p^0, Y^0) = h(p^0, C^0 + S^0)$$

$$(20) \qquad \tilde{q}^1 = h(p^1, Y^1) = h(p^1, C^1 + S^1)$$

represent alternative possibilities of choice for the consumer, where he has used also his actual saving S^0 and S^1 in buying consumer goods.

The purchasing power of income is connected with bundles (19)-(20), which the agent is able to buy with his income $Y = C + S$ in the two situations. Evidently, in the usual way of thinking the new income $Y^1 = C^1 + S^1$ is regarded to contain more purchasing power than the old income $Y^0 = C^0 + S^0$ if it can buy more consumer's goods, i.e. if $\tilde{q}^1$ lies on a higher utility level than $\tilde{q}^0$: $\tilde{q}^1 \quad \tilde{q}^0$ or $u(\tilde{q}^1) > u(\tilde{q}^0)$. This is in accordance with the definition of Keynes (1930, p. 54): "We mean by the Purchasing Power of Money the power of money to buy the goods and services on the purchase of which for purposes of consumption a given community of individuals expend their money income". An numerical measure of the purchasing power of Y^1 in relation to that of Y^0 is provided e.g. by the Paasche's type Allen quantity index

$$(21) \qquad Q(\tilde{q}^1, \tilde{q}^0; p^1) = \frac{C(p^1, u(\tilde{q}^1))}{C(p^1, u(\tilde{q}^0))} = \frac{Y^1}{C(p^1, u(\tilde{q}^0))}$$

This is a straightforward application of the theory of consumer's choice the calculations being made as if all income were consumed, and saving were identically zero. Just like in equations (6)-(7) the standard of living index (21) is the result of deflation, when the income ratio Y^1/Y^0 is deflated by the Laspeyres' type Konüs cost of living index:

$$(22) \qquad Q(\tilde{q}^1, \tilde{q}^0; p^1) = \frac{Y^1/Y^0}{P(p^1, p^0; u(\tilde{q}^0))} = \frac{Y^1/Y^0}{P_0^1} .$$

Thus $Q(\tilde{q}^1, \tilde{q}^0; p^1) > 1$ iff real new income Y^1/P_0^1 (which equals the real consumption C^1/P_0^1 plus real saving S^1/P_0^1) is greater than old income $Y^0 = C^0 + S^0$. This comes near to Usher's (1976) favourite definition of real income as the sum of real consumption and real saving.

This provides an easy and natural method to estimate the relative purchasing power of two incomes Y^1 and Y^0 because the cost of living index

$$(23) \qquad P(p^1,p^0;\ u(\tilde{q}^0)) = \frac{C(p^1,\ u(\tilde{q}^0))}{C(p^0,\ u(\tilde{q}^0))}$$

is easily approximated. In order to use (21) the bundles $\tilde{q}^1$ and $\tilde{q}^0$ and therefore the demand function h(p,Y) should be known. If P_0^1 is an approximation of (23) then, of course, an approximation of $Q(\tilde{q}^1,\tilde{q}^0;\ p^1)$ is

$$(24) \qquad Q_0^1 = \frac{Y^1/Y^0}{P_0^1}\ .$$

An excellent approximation P_0^1 of (23) would normally be

$$(24) \qquad P(p^1,p^0;\ u(q^0)) = \frac{C(p^1,\ u(q^0))}{C(p^0,\ u(q^0))}\ ,$$

where $q^0 = h(p^0,C^0)$ and $u(q^0)$ is the actual level of living bought by the old consumption expenditure C^0. The only difference between (23) and (24) is the utility level $u(\tilde{q}^0)$ and $u(q^0)$ which usually changes the cost of living index only marginally. If the preferences of the consumer are homothetic then (23) and (24) give exactly the same result.

A practical and fairly good approximation of both (23) and (24) is provided by the Laspeyres' price index (16). Its approximation error would be negligible if either the price changes are almost proportional, i.e. $p^1 \approx kp^0$ for some $k > 0$, or all the price changes $(p_i^1/p_i^0) - 1$ are small.

To be precise, (21) and (22) measure the purchasing power of income Y^1 in relation to Y^0 in the market for consumption goods. The addition is essential. All income is thought to be used for buying consumer's goods. If the purchasing powers of Y^1 and Y^0 are equal in the market for consumption goods, then $Q(\tilde{q}^1,\tilde{q};\ p^1) = 1$ and $Y^1/Y^0 = P(p^1,p^0;\ u(\tilde{q}^0))$.

Generally, if prices change, $p^0 \rightarrow p^1$, the consumption compensated income, or shortly C-compensated income $\bar{Y}^1$ satisfies

$$(25) \qquad \bar{Y}^1/Y^0 = P_0^1 \text{ or}$$

$$(26) \qquad \bar{Y}^1 = P_0^1 Y^0\ ,$$

where $P_0^1 = P(p^1,p^0;\ u(\tilde{q}^0))$. If the prices have increased $100P_0^1$ per cent then the C-compensated income $\bar{Y}^1$ must exceed Y^0 by the same relative amount in order to maintain its purchasing power in the consumption market.

This is the theoretical analysis of the ordinary and quite natural method of defining the purchasing power of income by deflation. Although it is deeply rooted in the theory of consumer's choice it has, however, some theoretical weaknesses and a more precise (but somewhat more complicated) method may be developed. The role of saving as a source of welfare must be analyzed more carefully. This is the task we will consider next.

3. THE EXPERIENCED UTILITY OF CONSUMING AND SAVING

Let's denote by

(27) $V(p,C,S;\ X)$

the (somewhat loosely defined) concept of (indirect) utility connected with prices of consumption goods p, total consumption expenditure C, total saving $S = Y - C$ and other relevant variables X (such as personal characteristics, attitudes, health, social relations, wealth, employment, interest rates, price expectations etc.) of the experiencing 'pleasure machine', e.g. person or household. Although 'other relevant variables' X may be of great significance to the welfare of our pleasure machine, these influences are ordinary ignored from the analysis by assuming that X remains practically constant. In that case (27) is shortened to $V(p,C,S)$. However, we must occasionally discuss the role of 'other relevant variables' in order to draw a line between the welfare effects to be included in the analysis and other more general welfare effects (say of health, social relations, or wealth) which are not analyzed in detail. An alternative concept is the (more or less direct) utility function $U : \mathbb{R}^{2n}_{++} \times \mathbb{R} \to \mathbb{R}$ defined by

(28) $U(h(p,C),p,S) := V(p,C,S)$.

More intuitively $U(q,p,S)$ gives the utility corresponding to the consumption bundle $q = h(p,C)$, their prices p and saving S.

Because of the importance of consuming (when X has been fixed) we may say, that q is the important variable in $U(q,p,S)$, while p and C mostly determine $V(p,C,S)$. We, so to speak, try to add the welfare effects of saving to the standard utility theory, where either the direct utility $u(q)$ related to the consumption of q or the indirect utility $v(p,C) = u(h(p,C))$ related to the consumption possibilities given indirectly by p and C are considered.

We describe the welfare experienced when a certain bundle of goods $q = h(p,C)$ is consumed during a given period and a given sum of money $S = Y - C$ (as a rule $S > 0$, but $S \leq 0$ is also possible) is saved, i.e. left over to be used later. We do not analyze whether our agent has any spesific plans concerning the future use of S or in what form it is restored. The details of the future are unknown and unsure to our agent, which is relying on his or her expectations about the future course of events. The allocation of income Y between consumption expenditure C and saving S must be based on comparing the <u>current</u> satisfaction of consuming C and the <u>current</u> satisfaction of not consuming S now but later[1]. The current satisfaction of saving consists mainly of the knowledge that S may be consumed later, in some way or another. Saving is <u>expected</u> to produce welfare in the future, but this mental process of expecting welfare in the future is a form of <u>current</u> welfare. The mere fact that people do not consume all their income nor their wealth now (although it is a possibility) shows, that they do not find

1) We suppose, unless otherwise stated, that S is positive which is the normal case. Negative saving or dissaving may be treated analogously.

it worth while doing so. The act of saving usually a part of their income is considered a better choice, i.e. a choice producing more satisfaction now. Only with children and in some expectional states of mind caused e.g. by heavy drugs or mental diseases the time horizon seems to shorten practically to zero.

The composition of saving S need not be represented in the model. Saving may occur in the form of various assets in banks or savings and loan associations, as purchases of bonds or obligations, payments of insurance premiums, repayments of debts, etc. If $S = S_1 + \ldots + S_K$, where S_k's are different components[1] of S then both S and all S_k's may be expressed as functions of prices p, income Y and "other relevant variables" X. In the allocation of S into $S_1, \ldots, S_k$ the role of "other relevant variables" X will be crucial. But in allocation of Y into C and S only an overall knowledge of the economic situation and a rough description of our agent's intentions are necessary.

The relative utilities of different forms of saving S_k depend on their yields, liquidity, safety etc. But their relative popularity is also strongly dependent on the economic knowledge of our agent: an ordinary saver doesn't buy shares, bonds or gold although a more economically educated saver could. Lack of information is here a more pronounced aspect than in pure consumer theory, because such special information is not generally needed in buying consumer's goods, perhaps with the exception of some highly developed consumer durables.

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1) As we will argue later there are even logical difficulties to decompose S into $S_1, \ldots, S_K$; in other words the representation $S = S_1 + \ldots + S_K$ must be highly arbitrary. Therefore such decompositions shouldn't be regarded very interesting nor important.

The real difficulty in explaining saving compared to explaining consumption is that the consumption of goods is much more easily quantified in terms of physical or other suitable units. On the other hand only the money value of saving is ordinarily estimated with some precision while the physical units of different forms of saving are as a rule lacking. Therefore we cannot construct any direct utility function for saving (as we can in the case of consumption) but indirect utilities need to be applied instead. In short: the generalization of utility theory to include also saving decisions must be carried out via indirect utilities.

There are other attempts to generalize consumer theory to include also saving decisions. The first idea is to deflate the saving S by some "price index" P and calculate "real saving" as S/P which is added as an argument to the direct utility function $u(q) = u(q_1, \ldots, q_n)$. The generalized utility of consuming a quantity bundle q and saving S money units is here taken to be U(q, S/P), where the "real saving" S/P is treated in a similar way as any physical quantity q_i. The unsolved difficulty of this approach is how the "price index" P should be defined. Usually it is some exogenous function of prices, say $P = P(p) = \Sigma c_i (p/p_i^0)/\Sigma c_i$, where $c_i > 0$ and $p_i^0 > 0$ are exogenous constants.

In this approach the definition of "real saving" is carried out outside the theory (exogenously) as in our attempt the right "deflator" of S (so to speak) is derived within the theory.

It is not our aim to deny the intuitive content of this mathematically straightforward generalization U(q, S/P) of the ordinary utility function u(q), but this theory seems to be too rigid and special to provide a systematic starting point for discussions about the compensated income in consumption-saving economy.

It will be an interesting task to compare the implications of U(q, S/P)-theory and our V(p,C,S)-theory, which we shall leave to the proponents of the former theory.

The second and more widely used idea to generalize consumer theory is the intertemporal theory, where in addition to the 'current' period t also one or more future periods t+1, t+2,... are considered. In the two period model $U(q^t, q^{t+1})$ is used as a generalization of the one period utility function u(q), or in fact $u(q^t)$. In the intertemporal theory current and future consumption decisions are determined by the same maximization of $U(q^t, q^{t+1})$ subject to a generalized budget or wealth constraint. By saving (or dissaving) income is transformed from one period to another. Intertemporal preference theory is a beautiful but complicated mathematical theory, see e.g. Strotz (1955), Pollak (1968), Koopmans (1970), Blackorby, Primont and Russel (1978, p. 341-356). Pollak (1968, 1975) considers different definitions of the true cost of living index in this framework, but there is no concensus how the cost of living index and the purchasing power of income should be calculated in the intertemporal theory. These difficulties of the intertemporal theory arise because an optimum for some period depends as a rule of data from all other periods.

The great theoretical difficulty of the intertemporal preference theory is the once-and-for-all-determination of the 'optimal' consumption plan and the corresponding (life time) income allocation. This is also an indication of its unrealism. The future is treated as if the economic agent had a perfect foresight and knowledge of all relevant future events, e.g. of prices and of his preferencies. Because this is not (fortunately!) the case people do not even try to determine their future consumption plan in all its details during any planning period. Because of its unrealism and mathematical complexity we have not taken the intertemporal theory as our starting point. Our aim is to develop a new V(p, C,S)-theory, which is more general than U(q, S/P)-theory, but simpler and, as we see it, more realistic than the intertemporal theory . Philosophically V(p,C,S) is an expression of the estimated experienced welfare now resulting from consuming C and saving (dissaving) S money units under prices $p = (p_1,...,p_n)$.

Sometimes it is argued that the structure or composition of saving should be taken into account in the estimation of the welfare effects of saving or its price index. It may be argued e.g. that the (expected) price development of say stocks and forests (land) may differ considerable or that saving used for repayment of unindexed loans does not need compensation in an inflationary situation while e.g. investment in stocks usually does. These comments may be valuable in the allocation of wealth or in choosing the optimal asset portfolio but they are to a large extend irrelevant and even impossible to take into account in analyzing the welfare effects of saving. They are symptoms of an insufficient separation of accumulated savings or wealth from saving, which is a common source of confusion. To make our point clear we would like to stress that the impossibility to disaggregate the saving (i.e. the difference of disposable income and consumption) into different components or uses is not a practical difficulty caused by the lack of suitable data but a logical impossibility. There does not exists any such thing as "saving disaggregated to different uses". This is mere misuse of words.

To give an example, consider periods of increasing length starting from a given point of time. Suppose that saving during the first five months is positive, say 20 % of income, and imagine for a while that it is decomposed in come way to "its different uses". During the next month consumption exceeds the monthly income considerably (e.g. because of vacation) so that total saving during the first half a year drops to 5 % of income. How should the negative saving of the last month be deducted from the "different uses of saving" of the first five months to get the decomposition of saving for the first half a year? To our knowledge no method of decomposition has been nor can be presented. To demonstrate the difficulty even more forcefully, suppose that during the seventh month a piano is bought which turns total saving negative. We will continue to regard the decomposition impossible unless someone presents a logical and operational way of decomposing saving to its different uses during any observation period.

We are not arguing that this will hinder us to speak of different motives of saving. According to Keynes (1936, p. 107-8) there are, in general, eight main motives or objects of a subjective character which lead individual to refrain from spending out of their incomes. Keynes calls them the motives of Precaution, Foresight, Calculation, Improvement, Independence, Enterprise, Pride and Averice. These motives describe the process of saving and consuming in general terms, not the individual decisions, and they are directed to keep the stock of wealth in harmony with the individual preferencies.

Although individual saving as a fluctuating difference of the flow of income and the flow of consumption cannot be objectively divided into different uses it comprises together with changes in valuation the change in Net Wealth. The Net Wealth equals Material Wealth plus Claims minus Liabilities, the three of which have, of course, natural subdivisions according to their "uses". It would be a very ambitious and difficult task to try to derive consumption and saving decisions from a utility model including as well the optimal allocation of wealth. Our aims are not so ambitious but the wealth side is left unanalyzed. We deliberately refuse to take into account in our model the welfare effects of changing wealth portfolio but concentrate on the welfare derived from the disposable income of our decision maker. Thus we will concentrate on the welfare experienced by the decision maker in the role of an income user, which is just the role of a consumer-saver.

It must be admitted that by concentrating in this way on certain effects, which on the other hand implies exclusion of some other phenomena, some perplexing conceptual difficulties arise. One may wonder how saving could be analyzed without simultaneously analyzing the changes in wealth it necessarily produces. It is difficult to decide where to draw the line between the things which are and are not analyzed in a particular theoretical model, but anyhow the line must be drawn somewhere. This is a necessary methodological choice in analytic science, which separates it e.g. from all-inclusive religious thinking or oriental philosophies. A large part of analytical economics is in fact based on such a theoretical conception or restriction, namely on the concept of homo economicus, see Pareto (1971, chapter 1) and Machlup ( ). Therefore we do not apologize any more our deliberate concentration to analyze the homo economicus (a role itself) only in one of its particular roles, namely in the role of an income user, which is a concentration we are not only allowed to make but which in a form or another is a necessity. We so to speak try to extend the analysis of the role of a consumer to that of an income user, which means that the two conflicting roles of a consumer and saver are analyzed together. To delineate our analysis e.g. the roles of a labourer (or income earner) or of a wealth owner (a capitalist) are not considered explicitely in any detail in our model. Variables connected in these roles are given exogenously. We cannot, however, omit the role of a capitalist entirely, because every nonzero saving brings about a change in the wealth portfolio. How this change is realized is left untold.

We admit that some kind of change in the wealth portfolio occurs because of (positive or negative) saving and the welfare effects of this change are calculated on the "utility account" of our income user. In the most simple and common possibility when the lenght of the planning period is a year or less saving affects only bank deposits or some other liquid sources of money and other forms of wealth are left unchanged. In calculating the welfare of our income user in this basic case only the net effects of bying consumer goods by consumer expenditure C and changing liquid sources of money by $S = Y - C$ (which may be negative) are taken into account. If $S = 0$ all the welfare is produced by consuming $Y = C$. But we allow also more fundamental changes in the wealth portfolio during the planning period. Consider the possibility that our decision maker buys an apartment spending most of his previous savings and a new bank loan. This diminishes his liquid money reserves and increases his material wealth and liabilities. At the some time, however, saving $S = Y - C$ from actual income Y during the planning period may be either positive or negative depending on the consumption C. If saving S is positive (negative) the liquid money reserves may be thought to increase (decrease) by the same sum (from what it would have been if S were zero) and all the other wealth components are kept unaffected. Here we think that the apartment would have been bought regardless of the value of saving S during the planning period and therefore also here saving may be thought to affect only liquid money reserves.

This is the way we will separate the decisions concerning the allocation of wealth from/that of allocating income to consumption and saving. Saving S is considered to affect only liquid money reserves and all other changes in the wealth portfolio during the planning period are considered as exogenously given. The welfare effects of saving S during the planning period from the point of view of our income user are calculated on the basis of this hypothesis.

We shall illustrate the roles of a consumer, saver and capitalist and their respective utilities using a simplified utility model. Let's denote by C and S consumption expenditure and saving, by p the price vector of consumption commodities and by $W_M = p_M \cdot q_M = \Sigma\, p_{Mi} q_{Mi}$, $W_{CL} = p_{CL} \cdot q_{CL}$, $W_L = p_L \cdot q_L$ the value of material wealth, liabilities and claims all expressed in current prices. Their quantities and prices are given by vectors q_M, q_{CL}, q_L and p_M, p_{CL}, p_L expressed in conventionally chosen units or in money terms.

We suppose that (C, S, q_M, q_{CL}, q_L) assume their optimal values and the combined utility of the income user (or consumer-saver) and capitalist is given for purpose of illustration by the following expression:

(29) $$K(p, C, S, p_M, p_{CL}, p_L, q_M, q_{CL}, q_L)$$

$$= [C+S]/P(p) + [W_M + W_{CL} - S - W_L]/P(p)$$

$$= [C+S]/P(p) + [p_M \cdot q_M + p_{CL} \cdot q_{CL} - S - p_L \cdot q_L]/P(p).$$

Here $W = W_M + W_{CL} - W_L$ is the average value of the net wealth during the planning period (not the starting or end value), $W - S = W_M + W_{CL} - S - W_L$ is the average 'operating net capital' of the capitalist and P(p) is the value of the consumer price index the form of which depends on the system of preferencies. The price index P(p) is used to deflate current values into real terms: (C+S)/P(p) is the real income or the purchasing power of the income which is an expression of the utility of our income user (or consumer-saver) and $(W-S)/P(p) = [W_M + W_{CL} - S - W_L]/P(p)$ is the real value of the operating capital of the capitalist, which is at the same time an expression of the utility experienced in the role of a capitalist. To avoid double counting of saving S it is deducted from the net wealth W to get the operating capital of the capitalist W-S and the utility corresponding to it.

Consider now two different developments of the world during the planning period and the corresponding optimal values of the strategic variables $(p^0, C^0, S^0, p_M^0, p_{CL}^0, p_L^0, q_M^0, q_{CL}^0, q_L^0)$ and $(p^1, C^1, S^1, p_M^1, p_{CL}^1, p_L^1, q_M^1, q_{CL}^1, q_L^1)$. The change in the combined utility of the income user and the capitalist is

(30) $$\Delta K = K(p^1, C^1, S^1, p_M^1, p_{CL}^1, p_L^1, q_M^1, q_{CL}^1, q_L^1) - K(p^0, C^0, S^0, p_M^0, p_{CL}^0, p_L^0, q_M^0, q_{CL}^0, q_L^0)$$

$$= \Delta\left[\frac{C+S}{P(p)}\right] + \Delta\left[\frac{W_M + W_{CL} - S - W_L}{P(p)}\right]$$

expressed in obvious notation. Here

(31) $$\Delta \frac{C+S}{P(p)} = \frac{C^1+S^1}{P(p^1)} - \frac{C^0+S^0}{P(p^0)}$$

$$= \frac{C^1}{P(p^1)} - \frac{C^0}{P(p^0)} + \frac{S^1}{P(p^1)} + \frac{S^0}{P(p^0)}$$

$$= \Delta \frac{C}{P(p)} + \Delta \frac{S}{P(p)}$$

is the change in the utility of the income user, which partitions into changes of the utilities experiences in the roles of a consumer $\Delta \frac{C}{P(p)}$ and a saver $\Delta \frac{S}{P(p)}$. Similarly

(32) $$\Delta [\frac{W_M + W_{CL} - S - W_L}{P(p)}]$$

$$= \Delta \frac{W_M}{P(p)} + \Delta \frac{W_{CL}}{P(p)} - \Delta \frac{S}{P(p)} - \Delta \frac{W_L}{P(p)}$$

is the change in the utility of the capitalist. Suppose for instance that the prices of material capital goods differ in the two developments of the world, $p_M^0 \neq p_M^1$, but that the portfolio of material capital goods remains unchanged, $q_M^0 = q_M^1$. Then the term

(33) $$\Delta \frac{W_M}{P(p)} = \frac{W_M^1}{P(p^1)} - \frac{W_M^0}{P(p^0)}$$

$$= \frac{p_M^1 \cdot q_M^1}{P(p^1)} - \frac{p_M^0 \cdot q_M^0}{P(p^0)}$$

$$= \sum_i (\frac{p_{Mi}^1}{P(p^1)} - \frac{p_{Mi}^0}{P(p^0)}) q_{Mi}^0$$

gives the change in the utility of the capitalist caused by the changes in the valuation of the material capital goods, say changes in the prices of apartments or forests. This kind of capital gains or losses may be considerably greater than e.g. the change $\Delta \frac{C}{P(p)}$ in the real value of consumption expenditure. Therefore it is most essential that the factors affecting the welfare of the capitalist are clearly separated from those affecting the welfare of the income user (or consumer-saver). In order to keep the welfare of the income user constant his income $Y^1 = C^1 + S^1$ must be so determined that $\Delta \frac{C+S}{P(p)}$ equals zero. The resulting income Y^1 is the compensated income compensating for the change $p^0 \rightarrow p^1$ in consumer prices. At the same time the welfare of the capitalist may increase or decrease considerably but these changes are totally irrelevant and should not be taken into account when the welfare of the income user (or the purchasing power of the income) is the ultimate aim of the analysis. Similar comments concerning the exogenous treatment of the role of a labourer or an income earner (described e.g. by allocation of working time and leisure, choice of job, monthly salary and other components of income etc) could be given. Exogenous treatment of these factors is usually, however, regarded as nonproblematic because they are traditionally considered as exogenous in the consumer theory. The problems in the separation of the role of an labourer from that of an income user are in principle neither smaller nor greater than the previously described difficulties to separate the roles of an capitalist and an income user.

We have used as our starting point the general decision theoretic framework presented by Törnqvist and Nordberg (1968). They stress the importance of recognizing that the although future is unknown to us, the decisions must be made on the basis of the information and preferencies the decision maker has 'now', i.e. in the actual decision situation.

Törnqvist and Nordberg (1968, p. 11) say that a decision process is genuine if it contains the following five stages:

1. Investigation of the decision situation, i.e. sharpening the view of history and the information it contains
2. Clearing up the various decision possibilities.
3. Shaping the views of future corresponding to various decision possibilities.
4. Valuation and comparison of the views of future corresponding to various decision possibilities.
5. Choice of the decision from varioud decision possibilities.

The realization process is divided into two parts:

6. Actual realization of the decision.
7. Collection of information about the consequences of the realized decision.

Although a decision is formally a choice from various decision possibilities (stage 5.) it is based on the expected consequencies ("the view of future") corresponding to the decision (stage 3.) and on their subjective valuation (stage 4.). The views of future corresponding to various decision possibilities and their valuation are dependent on the decision maker and on the decision situation. Often only the decision possibilities (e.g. the consumption bundle q, consumption expenditure C and saving S) are presented in the formal model describing the decision process while the views of future corresponding to the various decision possibilities are left untold. In that kind of formal framework the decision possibilities are valued and compared without explicite reference to their expected consequencies using for instance a utility function of the decision situation defined on the space of different possibilities (say $u(q) = u(q;X)$, $U(q,p,S) = U(q,p,S;X)$ or $V(p,C,S) = V(p,C,S;X)$, where the often omitted vector valued X-variable gives a more detailed description of the decision situation). Even though the expected consequencies or the views of future corresponding to various possible decisions are omitted from the formal decision model (evidently because of the utmost complexity to describe them) they must be kept in mind when building the theory. By doing so we may successfully generalize the classical utility theory to include saving decisions and perhaps see more clearly some of its limitations.

By the indirect utility function V(p,C,S) we intend to spesify for any fixed price vector the totality of (C,S)-pairs producing any preassigned level of utility as experienced now by our agent. Let us investigate the possible forms of these isoquants for an agent planning the use of his or her income during the next period of a month's length[1], say. It is not our aim to present a psychologically accurate theory of decision making but to describe the economic consequencies of expected decisions during the planning period. Although decisions are made sequentially rather than as one overall decision, a reasonable view is achieved by considering the sequence of decisions during the period as one superdecision. It may be thought that the <u>average</u> welfare during the period is related to the <u>totality</u> of purchases of goods and to other conditions during the period[2].

In a neat case the isoquants of V(p,C,S) in (C,S)-space for a fixed p-vector look as shown in figure 1.

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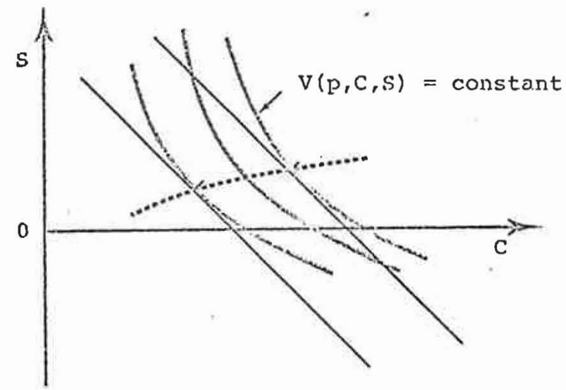
1) The length of the planning period is an interesting parameter in the problem. Apparently, people do not have any planning period of prefixed length but its length depènds on the situation. Food purchases are usually made in urban surroundings in view of a few days' consumption but purchases of consumer durables or summer trips are planned as a rule several weeks in advance. The actual purchases could be described as a chain of decisions based on daily (or hourly?) utility functions which adjust to actual purchases or fulfillment of wants. E.g. having bought a new jacket diminishes the desire to buy another for a while thus changing the utility function.

2) In order to explore the agent's preferences by interviewing also the agent needs some imagination and knowledge of economics because in making essential questions most details of practical decision making must be idealized away.

<u>Figure 1:</u> Possible isoquants of V(p,C,S), when p is given



The isoquants are convex[1] and each of them osculates some budget line C + S = Y at a unique point $(\hat{C},\hat{S})$, where saving $\hat{S}$ is usually positive. Dually, for any given income Y there exist exactly one point $(\hat{C},\hat{S})$ which maximizes V(p,C,S) for a given p. Therefore optimal $\hat{C}$ and $\hat{S}$ are here unique functions of prices p and income Y: $\hat{C} = \hat{C}(p,Y)$, $\hat{S} = \hat{S}(p,Y)$.

We have also shown by the dotted line, how $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ change when income Y increases.

If our agent insists on buying a colour TV or having a vacation next month then his planned saving would probably be negative and the possible isoquants are presented in figure 2.

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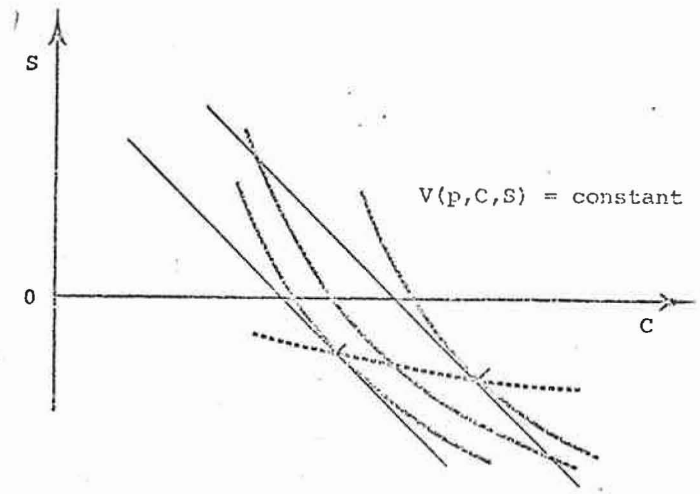
1) I.e. they are totally "above" any of their tangents.

Figure 2: Possible isoquants of V(p,C,S) for a given p when the agent has decided to have a vacation next month



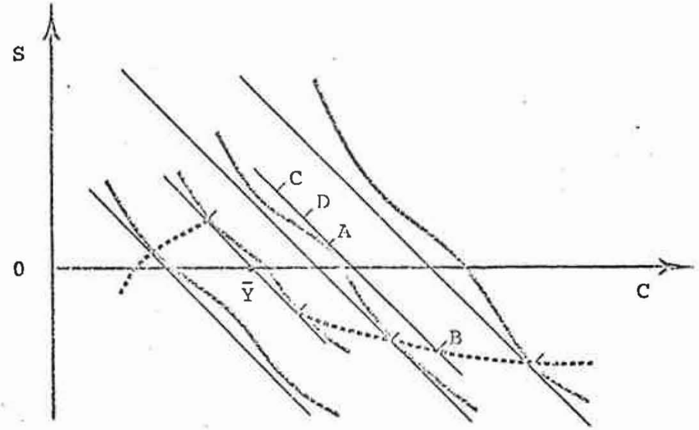
Optimum $\hat{C}(p,Y)$ exceeds income Y, while $\hat{S}(p,Y)$ is negative for moderate Y. Both are continuous functions of Y.

Note that the differences in figures 1 and 2 may be explained by 'other relevant variables' X, which may well include components describing the decision activity of our agent. Therefore by deciding to do something during the next month affects, of course, the preferencies between C and S.

If our agent has not yet decided have a vacation but wavers between having a vacation next month or later then his V(p,C,S) is a kind of mean between the corresponding utility functions shown in figures 1 and 2. The isoquants of this V(p,C,S) need not be convex but may have the shapes[1)] shown in figure 3.

1) Because any additional dollar either for consuming or saving is regarded as a benefit, i.e. $\partial V(p,C,S)/\partial C > 0$ and $\partial V(p,C,S)/\partial S > 0$, the isoquants cannot have vertical or horizontal segments.

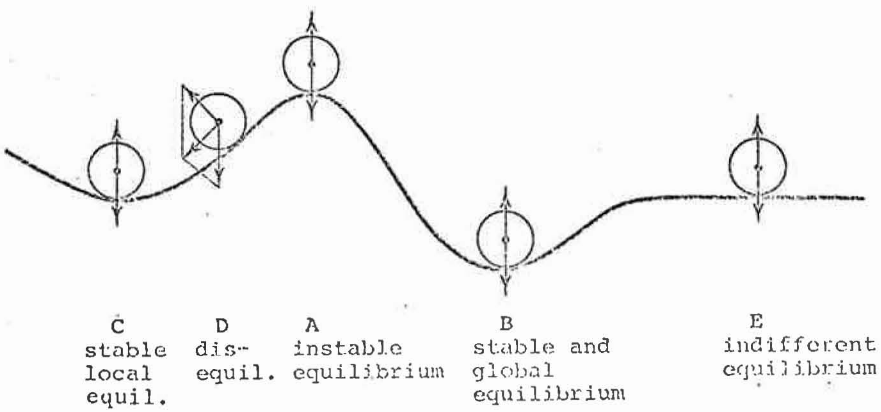
Figure 3: Possible isoquants of V(p,C,S) for a given p when the agent wavers between having a vacation next month or later



For high incomes the vacation possibility during the next month seems to be more attractive than ordinary life, but if incomes happen to be lower the vacation gradually looses its attractivity. For an income $\bar{Y}$ vacation and ordinary life are valued equally high and the optimum $(\hat{C},\hat{S})$ is not unique. For still lower incomes ordinary life is valued more attractive. Thus $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ may be even discontinuous for some Y, because the isoquants of V(p,C,S) are not convex in figure 3. Note also that even though in the point A the isoquant is tangent to the budget line, it is not an optimum choice. But by comparing only very small displacements on the corresponding budget line through A the agent may miss to recognize that his satisfaction would increase up to C and B, the latter being the optimum choice. This kind of peculiarities (which seem to reflect actual difficulties in real decision making) do not arise if the isoquants of V(p,C,S) are assumed to be convex as in figures 1 and 2. Note also that the point C is a local optimum from which it would be hard to recognize that a quite different 'way of life' represented by point B would give more satisfaction. But isn't this a realistic feature in the model representing the occational doubt in our minds that our present choice may be only the second best, while the best choice might require unpleasantly great changes.

These different "equilibrium" points may be visualized by a physical analog. In figure 4 the locations A, B and C of a ball correspond to points A, B and C in figure 3, while D is in both figures a point of disequilibrium.

Figure 4: A ball in different equilibrium or disequilibrium situations



All the equilibrium points (including the unstable equilibrium A) are found by setting the derivatives of the lagrangean expression

(34) $F(C,S,\lambda) = V(p,C,S) - \lambda(Y - C - S)$

equal to zero, which gives the first order conditions

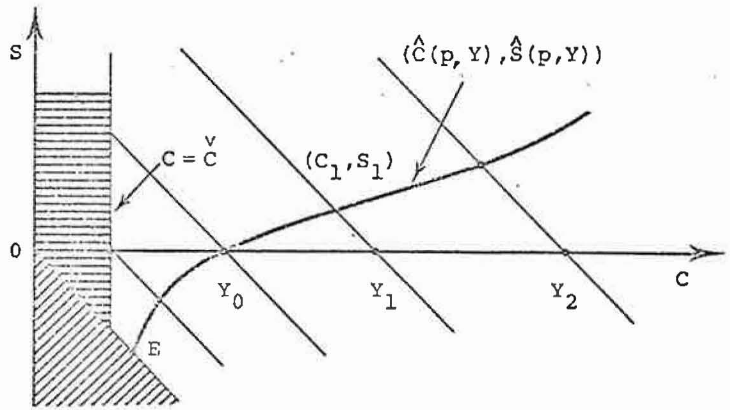
(35) $\partial V(p,C,S)/\partial C = \partial V(p,C,S)/\partial C$

(36) $Y = C + S$.

The tangency condition (35) says that the marginal utilities of consumption and saving should be equal in an equilibrium point. Optimum $(\hat{C},\hat{S})$ for a differentiable V(p,C,S) is a solution of (35)-(36) for which $V(p,\hat{C},\hat{S})$ is maximum. Even for nonconvex isoquants of V(p,C,S) the optimum point $(\hat{C},\hat{S}) = (\hat{C}(p,Y), \hat{S}(p,Y))$ is unique for almost all $(p,Y) \in \mathbb{R}^{n+1}_{++}$. For given prices p the functions $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ are continuous between some isolated Y-values, where two or more points $(\hat{C},\hat{S})$ give the same maximum of V(p,C,S). In figure 3 $\bar{Y}$ is such an isolated Y-value. For convex isoquants $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ are unique and continuous for all $Y > 0$. The functions $\hat{C} : \mathbb{R}^{n+1}_{++} \to \mathbb{R}_{++}$ and $\hat{S} : \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ are called the ordinary (or noncompensated) consumption and saving functions, respectively.

In an ordinary simple case of convex isoquants and positive saving for moderate incomes $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ for a given p change in the way shown in figure 5.

Figure 5: Optimum $(\hat{C},\hat{S})$ for a given p and different incomes Y in an ordinary simple case

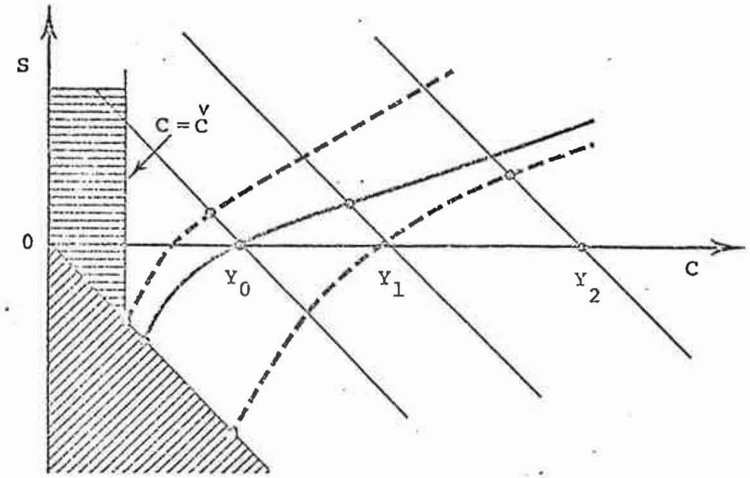


Here Y_1 is the expected (or permanent) income for the next month and our agent plans to consume C_1 and save $S_1 > 0$ for that income Y_1. Both consumption and saving will increase if Y exceeds Y_1 but saving would increase more rapidly.

For a smaller income Y_0 planned saving would equal zero and $\hat{C}(p,Y_0) = Y_0$. If income would be still smaller saving would turn negative. In point E income equals zero and consumption is financed completely by dissaving. Consumption in E still exceeds the necessary consumption denoted by $\check{C}$. The shaded area in figure 4 is considered inadmissible because of the negativity of income $Y = C + S$. For convenience, also $Y = 0$ is excluded from the domain of $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$.

The solid line is transformed from figure 5 to figure 6, while the dotted lines represent consumption and saving functions for a poorer and richer agent respectively.

Figure 6: Consumption and saving functions for a given p for three agents a_0, a_1 and a_2 having permanent incomes $Y_0 < Y_1 < Y_2$ respectively



The richer agent a_2 saves slighty less from his permanent Y_2 than a_1 would if he had exceptionally high income Y_2. If the income of the rich a_2 would be exceptionally low he would turn quickly into a dissaver and would consume for zero income much more than a_1. On the other hand the poor a_2 has permanent income equal to a small income Y_0 and would reduce his consumption almost to a necessary minimum if his income would fall to zero.

In the standard case of linear consumption function

$$\hat{C} = \hat{C}(p,Y) = a + bY \quad (37)$$

(where $a > 0$ and $0 < b < 1$ may depend on prices p) also the saving function is linear in Y,

(38) $$\hat{S} = \hat{S}(p,Y) = Y - (a + bY)$$

$$= -a + (1-b)Y, \quad 0 < 1-b < 1 .$$

Because from (32) $Y = (\hat{C} - a)/b$ the saving function is linear also in $c = \hat{C}$:

(39) $$\hat{S} = -a + (1-b)(\hat{C} - a)/b$$

$$= -a\left(\frac{1-b}{b}\right) + \left(\frac{1-b}{b}\right)\hat{C} .$$

Therefore the points $(\hat{C},\hat{S})$ for different incomes Y would make up a straight line in (C,S)-space starting from $(C,S) = (b, -a(1-b)/b)$ for $Y = 0$ and having a slope equal to $\partial\hat{S}/\partial\hat{C} = (1-b)/b > 0$.

Although the indirect utility function V(p,C,S) of consuming and saving is a natural starting point when the welfare effects of saving are analyzed, some other representations are also illustrative. These representations are derived from previous results by the help of variable transformations. A simple transformation, where income-consumption pair $(Y,C) = (C + S, C)$ is used instead of (C,S) as an argument in the indirect utility function is the following

(40) $$\psi^*(p,Y,C) := V(p,C,Y - C).$$

$\psi^*(p,Y,C)$ is thus defined to be the indirect utility corresponding to having income Y dollars and consuming C dollars of it under prices p. But instead of having C as an argument of $\psi^*(p,Y,C)$ a more easily interpreted function uses the propensity to consume $c = C/Y$,

(41) $$\psi(p,Y,c) := \psi^*(p,Y,cY).$$

Thus $\psi(p,Y,c)$ is the indirect utility in the situation specified by (p,Y,c), i.e. having income Y and consuming 100c percent of it under prices p. Vectors (p,C,S) and (p,Y,c) determine each other uniquely and the mapping $(p,C,S) \xrightarrow{f} (p,Y,c)$, $f = (f_1,f_2,f_3) : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}^n_{++} \times \mathbb{R}_{++} \times \mathbb{R}$, is given by

(42) $$f_1(p,C,S) = p = (p_1,\ldots,p_n)$$

(43) $$f_2(p,C,S) = C + S \qquad \text{(i.e. Y)}$$

(44) $$f_3(p,C,S) = C/(C + S) \qquad \text{(i.e. c)}$$

Formally, the mapping $(p,C,S) \xrightarrow{f} f(p,C,S) = (p,Y,c)$ is a coordinate transformation and its inverse $f^{-1} = g = (g_1,g_2,g_3)$ gives (p,C,S) in terms of (p,Y,c). We have

(45) $g_1(p,Y,c) = p$

(46) $g_2(p,Y,c) = cY$ (i.e. C)

(47) $g_3(p,Y,c) = Y - cY$ (i.e. S).

Using (45)-(47) the function $\psi(p,Y,c)$ is expressed by

$$\psi(p,Y,c) = V(p,cY,Y-cY) \tag{48}$$

$$= V(g(p,Y,c))$$

$$= (V\circ g)(p,Y,c)$$

or shortly $\psi = V\circ g$. Similarly

$$V(p,C,S) = \psi(p,C+S,\frac{C}{C+S}) \tag{49}$$

$$= \psi(f(p,C,S))$$

$$=(\psi\circ f)(p,C,S)$$

or shortly and accurately $V = \psi\circ f$.

The following figure shows a typical indifference map in both coordinate systems and the determinations of the optimum pair $(\hat{C},\hat{S})$ and the corresponding optimum $\hat{c} = \hat{c}(p,Y) = C(p,Y)/Y$.

Here the curve $c = F(Y) = \check{C}/Y$ corresponds to the minimum consumption vertical in the first part of figure 5. The optimum propensity to consume satisfies $\hat{c}(p,Y) = \hat{C}(p,Y)/Y$ and has typically the shape shown in the figure. It is a slowly changing function of Y and may be derived also by maximizing $\psi(p,Y,c)$ for a fixed p and Y. The composite rule of differentiation gives

$$\partial\psi(p,Y,c)/\partial c = \partial V(g(p,Y,c))/\partial c \tag{50}$$

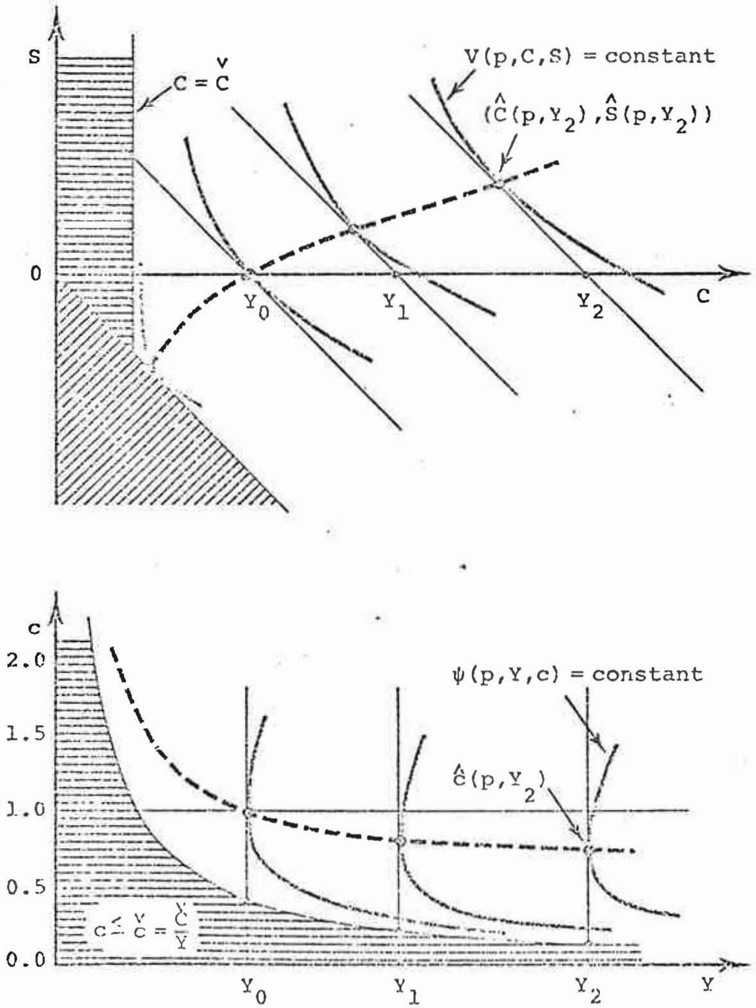
$$= \sum_{i=1}^{n} V_i(g)\frac{\partial p_i}{\partial c} + V_{n+1}(g)\frac{\partial g_2}{\partial c} + V_{n+2}(g)\frac{\partial g_3}{\partial c}$$

$$= \frac{\partial V}{\partial C}\frac{\partial cY}{\partial c} + \frac{\partial V}{\partial S}\frac{\partial(Y-cY)}{\partial c}$$

$$= \frac{\partial V}{\partial C}\cdot Y + \frac{\partial V}{\partial S}(-Y) = 0,$$

that is $\partial V/\partial C = \partial V/\partial S$, the former condition for equilibrium. We have thus shown, that in the optimum point $\hat{c} = \hat{c}(p,Y)$ we have $\partial\psi(p,Y,c)/\partial c = 0$, which means roughly that small changes in $\hat{c}$ changes the utility level $\psi(p,Y,\hat{c})$ only marginally. A useful approximation is that the utility $\psi(p,Y,c)$ of consuming 100c percent of Y is practically constant when $c \approx \hat{c}$. This fits to the general experience that people are rather indifferent about changing their propensity to consume some

Figure 7: Possible isoquants of $V(p,C,S)$ and $\psi(p,Y,c)$



percentage points (at least for a planning period of months length), which reflects their difficulty to make a clear distinction between consuming now or later. The force of this argument becomes less clear for small permanent incomes.

Note expecially that it is not true that for given prices and income by increasing the propensity to consume the agent would increase his welfare. In fact any deviation from his $\hat{c}$ would decrease the welfare although perhaps only marginally. This is in sharp contrast with consumer theory where all income must be consumed and the propensity to consume is necessarily unity.

4. EFFECTS OF PROPORTIONALLY CHANGING CONSUMER PRICES

This far we have kept prices p constant. In order to analyze effects of changing prices in consumption-saving economy we have to specify how $V(p,C,S)$ or $\psi(p,Y,c)$ are allowed to change when p varies.

Let all consumer prices change proportionally from p^0 to $\lambda p^0 = p^1$, where superscripts 0 and 1 denote two alternative situations for the same planning period. Consider solutions (C^1,S^1) of the equation

$$V(\lambda p^0,C^1,S^1;\ X^1) = V(p^0,C^0,S^0;\ X^0), \tag{51}$$

where other relevant variables X are also included for the two alternative situations. If other relevant variables X do not change much because of proportionate changes in consumer prices (or change in a way that does not influence the preferencies) then $(C^1,S^1) = (\lambda C^0,\lambda S^0)$ should be a solution of (51). This means that under λp^0-prices the pair $(\lambda C^0,\lambda S^0)$ and the income $\lambda Y^0 = \lambda C^0 + \lambda S^0$ would give the same utility for our income user as (C^0,S^0) and $Y^0 = C^0 + S^0$ would give under p^0-prices. This is a natural homogeneity assumption on the utility function $V(p,C,S;\ X)$. We will call a utility function $V(p,C,S,X)$ p-regular if under customary ceteris paribus clauses on X

$$V(\lambda p^0,\lambda C^0,\lambda S^0;\ X^1) = V(p^0,C^0,S^0;\ X^0) \tag{52}$$

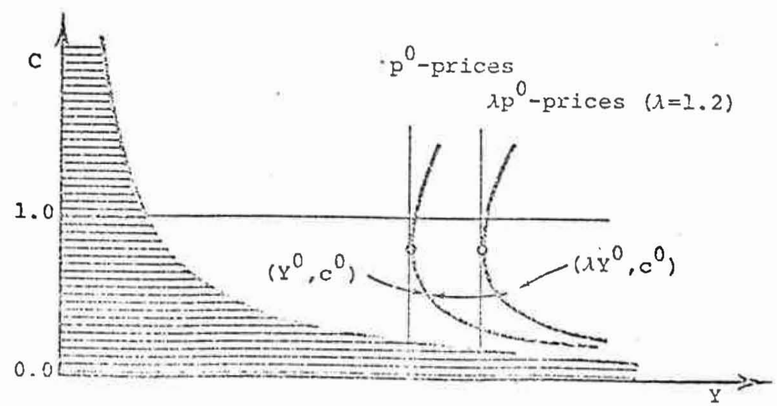
for all admissible (p^0,C^0,S^0) and $\lambda > 0$. This is a restriction for both the utility function and for the change $X^0 \to X^1$ in other relevant variables. The p-regularity assumption may be written for $\psi(p,Y,c;\ X)$ as follows:

$$\psi(\lambda p^0,\lambda Y^0,c^0;\ X^1) = \psi(p^0,Y^0,c^0;\ X^0) \tag{53}$$

for all admissible (p^0,Y^0,c^0) and $\lambda > 0$. This states that $(\lambda Y^0,c^0)$ gives the same utility under λp^0-prices as (Y^0,c^0) would give under p^0-prices whatever (Y^0,c^0) and $\lambda > 0$ are. We do not assume that (C^0,S^0) in (52) or (Y^0,c^0) in (53) are optimal pairs but they are any admissible pairs. Equations (52) and (53) generalize the homogeneity of zero property $v(\lambda p,\lambda C) = v(p,C)$ of an (ordinary) indirect utility function $v(p,C)$. More accurately this property should be stated as follows: For all admissible (p^0,C^0) and all $\lambda > 0$, $v(\lambda p^0,\lambda C^0;\ X^1) = v(p^0,C^0;\ X^0)$, where other relevant variables X have adjusted to the corresponding situations. This property or the corresponding property $h(\lambda p^0,\lambda C^0;\ X^1) = h(p^0,C^0;\ X^0)$ of the demand function is sometimes referred as the "absence of money illusion".

The p-regularity assumption may be illustrated using either of equations (52)-(53), but (53) is more convenient.

Figure 8: Two indifference curves of a p-regular utility function $\psi(p,Y,c;\ X)$ corresponding to the same utility and two price situations p^0 and λp^0



Here (Y^0,c^0) is any income-propensity-to-consume-pair and the left indifference curve gives all (Y,c)-pairs producing the same utility as (Y^0,c^0) under p^0-prices, i.e. for which $\psi(p^0,Y,c;\ \bar{X}^0) = \psi(p^0,Y^0,c^0;\ X^0)$. Other relevant variables (say consumption decisions) also adjust from X^0, when the income Y and the propensity to consume move on the indifference curve: $\bar{X}^0$ denots the adjusted X^0-values[1]. By the assumption of p-regularity $(\lambda Y^0,c^0)$ produces under λp^0-prices the same utility as (Y^0,c^0) produced under p^0-prices. The higher indifference curve contains by definition all (Y,c)-pairs satisfying

$$\text{(54)}\qquad \psi(\lambda p^0,Y,c,\bar{X}^1) = \psi(\lambda p^0,\lambda Y^0,c^0;\ X^1)$$

$$= \psi(p^0,Y^0,c^0;\ X^0)\ .$$

But in addition, if (Y,c) lies on the left indifference curve then $(\lambda Y,c)$ lies on the right curve:

$$\text{(55)}\qquad \forall (Y,c)\ :\ \psi(p^0,Y,c;\ \bar{X}^0) = \psi(p^0,Y^0,c^0;\ X^0) \Rightarrow$$

$$\psi(\lambda p^0,\lambda Y,c,\bar{X}^1) = \psi(\lambda p^0,\lambda Y^0,c^0,X^1) = \psi(p^0,Y^0,c^0;\ X^0)\ .$$

---

1) Similarly in the ordinary consumer theory the equation for an indifference curve should not be denoted by $u(q^0) = u(q)$ but by $u(q^0;\ X^0) = u(q;\ \bar{X}^0)$. Here X^0 describes among other things the view of future corresponding to the decision possibility q^0 and $\bar{X}^0$ contains similarly the view of future corresponding to q in the terminology of Törnqvist and Nordberg, cf. p. 29. The view of future and therefore X^0 must adjust to the change $q^0 \rightarrow q$ in the decision possibilities.

Therefore the right indifference curve is a λ-multiple expansion of the left indifference curve. If the same figure is drawn on a semilogarithmic paper (income Y plotted on the logarithmic scale) then the right indifference curve is merely a translation of the left curve.

Equations (52)-(53) hold, of course, under one change of money unit, where λ represents e.g. the exchange rate from marks to dollars. The change in the money unit is only a technical change in our way to describe the world, i.e. in our coordinate system: a change in the unit of measurement. The difficulty to distinguish the technical change in the money unit from an actual change in prices arises if other relevant variables X are omitted from analysis[1]. Therefore we use here e.g. $V(p,C,S;X)$ instead of customary $V(p,C,S)$.

It is impossible to give a complete list of those changes in X which do not violate the "customary ceteris paribus assumptions", but at least social conditions, health, general expectations and tastes of our income user should be kept unchanged. If tastes or plans of our income user change because of changing family relations (say marriage, divorce, birth of a child) or working conditions (change of job, unemployment) then (52) and (53) need not hold. Under

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1) For instance, Malinvaud (1972, p. 34) does not distinguish these cases, the "choice of numeraire" and "the absence of money illusion", and tries to derive results for the latter using arguments from the first.

changing tastes $(\lambda C^0, \lambda S^0)$ and $(\lambda Y^0, c^0)$ in (52) and (53) should be substituted by $(\tilde{\lambda} C^0, \tilde{\lambda} S^0)$ and $(\tilde{\lambda} Y^0, c^0)$ where $\tilde{\lambda}$ may be either greater or smaller than λ to make the utilities equal. Here $\tilde{\lambda}$ may depend in a complicated way of (p^0, C^0, S^0), λ and of the way X changes. Therefore this kind of changes in X are not allowed but are exogenized away by ceteris paribus assumption. Their effects require a separate analysis. Only those effects on X that result from proportionate changes in consumer prices are allowed here.

The question remains to what extent possible effects of proportionally changing consumer prices $p^0 \rightarrow \lambda p^0$ on other <u>economic</u> variables in X (e.g. prices of investment goods and assets, interest rate, total demand, employment etc) should be allowed and taken into account. If general consumer prices increase it causes an inflationary impulse raising prices of investment goods and wages; also real variables may be affected. These kind of predictable changes in X cannot be naturally assumed away by exogenizing the corresponding variables to their previous values.

It is a difficult and contraversial methodological question to what extent such effects of changing environment should be tried to be analyzed in the model and where the ceteris paribus clause should be applied. Anyhow, the pheonomenon (or system) under investigation should be separated somehow from other world but the line of demarcation often remains rather vague. Other researchers may want to include or

exclude certain considerations from the analysis. These are rather deep questions connected with the relationship between models and reality. In experimental sciencies the phenomenon under investigation is determined by describing the experiment, so that the cause variables, the response variables and the environment are clearly separated. For an expert of the field there are usually no problems what the ceteris paribus clauses are. In nonexperimental situations things are quite different. An analysis of the situation must be based on some <u>interpretation</u> or <u>view</u> of the situation; and as a rule more than one interpretation is possible. Therefore e.g. the cause variables, the response variables and the environment are usually defined in different ways in different interpretations - or these concepts are totally lacking. It seems to be here where the different "theoretical schools" of economics have their roots. For instance Pareto defends rather rigid forms of the ceteris paribus assumptions:

> "One is grossly mistaken then when he accuses a person who studies economic actions - or *homo oeconomicus* - of neglecting, or even of scorning moral, religious, etc., actions - that is the *homo ethicus*, the *homo religiosus*, etc. - ; it would be the same as saying that geometry neglects and scorns the chemical properties of substances, their physical properties, etc. The same error is committed when political economy is accused of not taking morality into account. It is like accusing a theory of the game of chess of not taking culinary art into account". (Pareto (1971) p. 13)

On the other hand Marshall has a much more synthetic view of economics:

"But ethical forces are among those of which the economist has to take account. Attempts have indeed been made to construct an abstract science with regard to the actions of an "economic man", who is under no ethical influences and who pursues pecuniary gain warily and energetically, but mechanically and selfishly. But they have not been successful, nor even thoroughly carried out. For they have never really treated the economic man as perfectly selfish: no one could be relied on better to endure toil and sacrifice with the unselfish desire to make provision for his family; and his normal motives have always been tacitly assumed to include the family affections. But if they include these, why should they not include all other altruistic motives the action of which is so far uniform in any class at any time and place, that it can be reduced to general rule? There seems to be no reason; and in the present book normal action is taken to be that which may be expected, under certain conditions, from the members of an industrial group; and no attempt is made to exclude the influence of any motives, the action of which is regular, merely because they are altruistic. If the book has any special character of its own, that may perhaps be said to lie in the prominence which it gives to this and other applications of the Principle of Continuity".
(Marshall (1920), Preface to the first edition).

More modern economic writers have a tendency to ignore these problems as if they were unimportant.

We suppose now that other prices $\bar{p}$ (including prices of investment goods, assets etc) change also proportionately ($\bar{p}^0 \rightarrow \bar{\lambda}\bar{p}^0$) because of change $p^0 \rightarrow \lambda p^0$.

Here $\bar{\lambda}$ may differ from λ. Even in that kind of situation equations (52) and (53) may well hold. Changing relative prices between consumer and capital goods causes, of course, capital gains or losses according to whether $\bar{\lambda} > \lambda$ or $\bar{\lambda} < \lambda$. These effects change the utility account of the capitalist

but should not be included to the utility account of our income user in order to avoid double counting. If for instance other prices remain unchanged[1] ($\bar{\lambda} = 1$) while consumer prices increase 10 % ($\lambda = 1.1$) the decision maker may suffer considerable capital losses in the role of a capitalist, because the value of the old capital goods corresponds to a smaller amount of more expensive consumer goods. Therefore the purchasing power of old capital has diminished about 10 %. However, compensated saving $\lambda S^0 = 1.1S^0$ buys 10 % more capital goods which have the same purchasing power during the planning period on the consumption market as the previous saving S^0 had under previous prices. Also the expected purchasing power of capital goods bought by $1.1S^0$ remains equal to that of S^0 under previous prices, if the relative prices between consumer goods and capital goods remain the same also in the future (p and $\bar{p}$ may both be constant in time or change in a predicted way).

But if prices of capital goods are expected to rise in the future in relation to consumer goods (if e.g. the 10 % gap between them is expected to vanish in three years) then saving during the planning period would be somewhat more

1) The assumption that other prices $\bar{p}$ remain constant although consumer prices p change is quite natural if p's are consumer spesific (not general) prices of consumption goods. These might change without affecting e.g. the general price level or other prices $\bar{p}$. In this micro economic context the assumption of p-regularity is also very natural, which shows that possible deviations from p-regularity are reflections of macro economic reactions.

profitable than previously because newly bought capital goods would increase their value in relation to consumer goods. In that case (C^0,S^0) need not be expanded 1.1-fold to attain the previous utility level but $(\tilde{\lambda}C^0,\tilde{\lambda}S^0)$ with $\tilde{\lambda} < 1.1$ would be sufficient. This is a special effect caused by a predicted difference between the rates of increase in consumer prices and other prices and in such a case the assumption of p-regularity need not hold exactly. Unless such special effects are expected $(\lambda C^0,\lambda S^0)$ may be safely used as a compensating (C,S)-pair producing the same utility under λp^0-prices as (C^0,S^0) produces under p^0-prices.

A similar argument holds when other prices are allowed to change, $\bar{\lambda} \neq 1$.

Thus when calculating the proportional expansion factors of consumption and saving (and thus of income) needed to compensate for a proportional change in consumer prices the natural assumption to start with is that other relevant economic variables in X change in such a way that the p-regularity assumption may be applied. Note also that if the p-regularity assumption is questioned in some situation then the expansion factor $\tilde{\lambda}$ in equivalent equations

$$(56) \qquad V(\lambda p^0,\tilde{\lambda}C^0,\tilde{\lambda}S^0;\ X^1) = V(p^0,C^0,S^0;\ X^0)$$

$$\psi(\lambda p^0,\tilde{\lambda}Y^0,c^0;\ X^1) = \psi(p^0,Y^0,c^0;\ X^0)$$

may be either greater or smaller than λ depending on the details of the situation. Therefore detailed information is needed in order to decide whether the p-regularity assumption should be rejected and in what direction it should be changed.

Another mathematically simple assumption with which our p-regularity assumption may be contrasted is the following: For all admissible (p^0,C^0,S^0) and $\lambda > 0$

$$(57) \qquad V(\lambda p^0,\lambda C^0,S^0;\ X^1) = V(p^0,C^0,S^0;\ X^0)\ .$$

Here it is assumed unrealistically that only consumption C needs compensation when consumer prices change proportionally. If e.g. $\lambda = 1.1$ (57) would say that $(1.1C^0,S^0)$ gives under 10 % higher consumer prices the same utility for our consumer-saver as (C^0,S^0) gave under previous prices p^0. This may hold only if S^0 is close to zero or other relevant variables X^0 have changed in some peculiar way, which has increased to profitability of saving. Vartia and Vartia (1979) argue that (58) is an implicite assumption in some occasionally made but unrealistic calculations concerning the burden of sales taxes.

As a summary, our problem has been which variables in X should be considered as endogenous and which should be exogenously fixed constants when effects of changing p is examined. Furthermore, for endogenous variables in X adjustments would be needed when p changes. An extreme view would be that all other variables X are exogenous constants (so that $X^1 = X^0$), in which case X could be omitted from analysis. This is probably the ordinary but dogmatic view adopted e.g. in consumer theory. (However, omission of X does not necessarily imply that this extreme view is adopted: possible changes in X and their effects are only left untold.) In our opinion a part of X must be considered as endogenous: as Dr. Pentti Vartia pointed humorously at least the price labels must change together with p. A more advanced analysis should therefore contain a system of <u>explicite behavioural equations</u> for some other endogenous variables. In our analysis these equations do not appear explicitely. However, their effects are allowed and taken into account implicitely, which will be a necessity in any empirical analysis for some kind of reactions. Thus any analysis will be "only" partial analysis but it will serve a useful purpose if its limitations have been specified.

5. CONSUMER DEMAND IN CONSUMPTION-SAVING ECONOMY

Here we will spesify how the income user allocates his consumption expenditures C among consumer goods. This has to be spesified not only for the optimal pair $(\hat{C},\hat{S})$ but for all admissible (C,S)-pairs, because we want to be also to compare any conceivable (C,S)-allocations and their implied suboptimal consumption patterns.

Let's start the analysis from the <u>behaviour</u> of our consumer-saver without any utility assumptions in the consumption space. We suppose that in any situation described alternatively by (p,C,S,X) or (p,C,c,X) the chosen unique consumption bundle $\hat{q}$ of the budget set $\{q \mid p \cdot q \leq C\}$ is given by a "demand function" $\bar{h}(p,C|S, X)$ or $h(p,C|c, X)$, respectively. These are two different functions which, however, determine each other uniquely. The latter $h(p,C|c; X)$ seems to be easier to work with. In accordance with the p-regularity assumption we suppose that the demand function is also homogenous of degree zero in (p,C):

<u>Homogeneity.</u> For all admissible variables the demand function $h(p,C|c,X)$ is homogenous of degree zero in (p,C) for any given (c,X): $h(\lambda p,\lambda C|c,\tilde{X}) = h(p,C|c,X)$,
where $\lambda > 0$ and $\tilde{X}$ is the adjusted value of X.
This means that in two situations $(\lambda p,\lambda C,c,\tilde{X})$ and (p,C,c,X), which amount to the same total utility by p-regularity, the chosen consumption bundle is also the same. In other notation, the two situations are described by $(\lambda p,\lambda C,\lambda S,\tilde{X})$ and (p,C,S,X),

which result in the following equivalent formulation using the demand function $\bar{h}(p,C|S,X)$:

Homogeneity. For all admissible variables the demand function $\bar{h}(p,C|S,X)$ is homogenous of degree zero in (p,C,S) for any given X: $\bar{h}(\lambda p,\lambda C|\lambda S,\tilde{X}) = h(p,C|S,X)$ where $\lambda > 0$ and $\tilde{X}$ is the adjusted value of X.
Here also saving must be rescaled in the new proportionate situation, $S \to \lambda S$, while the previous formulation propensity to consume remained constant, $c = C/(C+S) \to \lambda C/(\lambda C+\lambda S) = c$. This particularly makes $h(p,C|c,X)$ easier to work with than $\bar{h}(p,C|S,X)$.

We have not yet assumed that $h(p,C|c,X)$ has the properties of an ordinary demand function in (p,C), i.e. that $h(p,C|c,X)$ may be considered as a result of maximizing some utility function $u(q|c,X)$ under budget constraint $p\cdot q \leq C$. The consumer behaviour described by $h(p,C|c,X)$ might this far be of a more general type. But in that case only very limited inferencies concerning the consumer behaviour could be drawn and we couldn't in that case properly call our consumer-saver-theory a generalization of the ordinary consumer theory. Therefore we have to add some additional restrictions on the "demand function" $h(p,C|c,X)$ and indirectly on the utility function $\psi(p,Y,c;\ X)$.

We consider first the case $c = 1$ when all income is used in consumption. In order that our theory is a generalization of the ordinary consumer theory the function $\psi(p,C,1;\ X)$

should quality as an indirect utility function and the demand function $h(p,C|1,X)$ should be calculable from it by Roy's Theorem. We shall, however, start from some more basic assumptions which are more transparent in the general case when c may differ from unity. We say that the demand function $h(p,C|c,X)$ satisfies the weak consumer utility hypothesis (weak CUH) if the function $h(p,C|1,X)$ of (p,C) satisfies the utility hypothesis UH given before.

In other words if the propensity to consume is exogenized to unity, $c = 1$, while saving equals zero, the corresponding demand function $h(p,C|1,X)$ gives the unique maximum point $\hat{q}$ of some utility function $u(q|1,X)$ in the budget set $B(p,C) = \{q|p\cdot q \leq C\}$. The utility function $u(q|1,X)$ is assumed to fulfill conditions B so that the implied preference relation may be described as well using an indirect utility function $v(p,C|1,X) = \max_q\{u(q|1,X)|p\cdot q \leq C\}$. The demand function may be straightforwadly calculated from it by Roy's Theorem: $h(p,C|1,X) = h(r,1|1,X) = \nabla v(r,1|1,X)/r\cdot\nabla v(r,1|1,X)$, where $r = p/C$. Therefore the weak CUH we may be expressed dually also as follows:

Weak CUH: There exists an indirect utility function $v(p,C|1,X)$ satisfying conditions $\bar{A}$ and the demand function satisfies $h(p,C|1,X) = \frac{\nabla v(r,1|1,X)}{\nabla v(r,1|1,X)\cdot r} = C\frac{\nabla v(p,C|1,X)}{\nabla v(p,C|1,X)\cdot p}$ for all admissible arguments.

The indirect utility function $v(p,C|1,X)$ is determined only up to an increasing transformation so that it may be replaced by $v^*(p,C|1,X) = f(v(p,C|1,X))$, where $f: \mathbb{R} \to \mathbb{R}$ is any (differentiable) increasing function. Because $\partial v^*(p,C|1,X)/\partial p_i =$

$f'(v(p,C|1,X)\partial v(p,C|1,X)/\partial p_i$ also $v^*(p,C|1,X)$ leads to our previous demand function: $C\nabla v^*(p,C|1,X)/p\cdot\nabla v^*(p,C|1,X) = C\nabla v(p,C|1,X)/p\cdot\nabla v(p,C|1,X) = h(p,C|1,X)$ because $f'(v)$ cancels away.

Consider next the relationship between $v(p,C|1,X)$ and $\psi(p,C,1;\ X)$. Here $v(p,C|1,X)$ is an expression of the maximum consumption utility attainable in the situation $(p,C,1;X)$, while $\psi(p,C,1;X)$ is an expression of the maximum consumption-saving utility attainable in the same situation. But because saving is here forced to zero all the utility arises from consumption and both functions must tell essentially the same story. More exactly, both the utility functions must order different (p,C)-pairs in the some order: for all admissible (p,C) and $(\bar{p},\bar{C})$

$$(58)\qquad v(p,C|1,X) \geqq v(\bar{p},\bar{C}|1,\bar{X}) \quad \text{iff}$$

$$\psi(p,C,1;X) \geqq \psi(\bar{p},\bar{C},1;\bar{X})\ .$$

It is shown in Appendix 1 that this holds if and only if $\psi(p,C,1;X)$ is an increasing transformation of $v(p,C|1,X)$: $\psi(p,C,1;X) = g(v(p,C|1,X))$. Of course this holds also other way round, $v(p,C|1,X) = g^{-1}(\psi(p,C,1;X))$, because $\psi = g(v)$ and $v = g^{-1}(\psi)$ are both increasing.

We have thus shown that our (ordinal) consumption-saving utility $\psi(p,C,1;X)$ considered as function of (p,C) qualifies also as an ordinal consumption utility from which the demand function may be derived by Roy's Theorem. This is stated as a theorem.

Theorem 1. Suppose that $h(p,C|c,X)$ satisfies weak CUH with an implicite indirect utility function $v(p,C|1,X)$ satisfying (58). Then $v(p,C|1,X)$ and $\psi(p,C,1;\ X)$ are increasing transformations of each other both satisfying conditions $\bar{A}$ of an indirect utility function and the demand function $h^i(p,C|1,X)$ for any commodity a_i satisfies $h^i(p,C,1,X) = C\dfrac{\partial\psi(p,C,1;\ X)/\partial p_i}{\Sigma\ p_j\partial\psi(p,C,1;\ X)/\partial p_j}$

The weak CUH assumes that our income user behaves as an ordinary consumer if his saving happens to be or is exogenized to zero. In some connections we will make use also of a stronger assumption that our income user allocates always his consumption expenditure $C = cY$ among the consumer goods according to the ordinary consumer theory. This restriction on $h(p,C|c,X)$ is called consumer utility hypothesis:

CUH: The demand function $h(p,C|c,X)$ satisfies for any admissible c the utility hypothesis UH.

Denote the implicitely defined direct and indirect utility functions by $u(q|c,X)$ and $v(p,C|c,X)$. These function satisfy conditions B and $\bar{A}$ respectively for any values of c and

$$(59)\qquad h(p,C|c,X) = C\,\frac{\nabla v(p,C|c,X)}{\nabla v(p,C|c,X)\cdot p}$$

by Roy's Theorem. It remains to be described how the ordinal indirect utility functions of the consumer $v(p,C|c,X)$ and the income user $\psi(p,Y,c;\ X)$ are related to each other. Both functions are specified only up to an increasing transformation. We need to consider only cases where the propensity to consume c is fixed in both functions to a given value. Because $C = cY$

we may express Y as a function of C and c, $Y = C/c$ and consider the function

(60) $\bar{v}(p,C|c,X) = \psi(p,C/c,c;\ X)$

giving the total utility $\psi(p,Y,c;\ X)$ of the income user as a function of p, C and c.

Consider as an example the case $c = 0.8$ and two price-consumption situations (p,C) and $(\bar{p},\bar{C})$. Suppose that (p,C) is at least as good as $(\bar{p},\bar{C})$ from the point of view of the consumer: $v(p,C|0.8,X) \geq v(\bar{p},\bar{C}|0.8,\bar{X})$. More accurately, the price-consumption situation (p,C) and the propensity to consume $c = 0.8$ is valued at least as good as the pair $(\bar{p},\bar{C})$ with the same propensity to consume. Therefore we must also require that the income user orders the situations in the same way: $\bar{v}(p,C|0.8,X) \geq \bar{v}(\bar{p},\bar{C}|0.8,\bar{X})$, i.e. $\psi(p,C/0.8,0.8;\ X) \geq \psi(\bar{p},\bar{C}/0.8,0.8,\bar{X})$. The same must hold for any c, so that we require that for any admissible pairs (p,C) and $(\bar{p},\bar{C})$

(61) $v(p,C|c,X) \geq v(\bar{p},\bar{C}|c,\bar{X})$ iff

$\bar{v}(p,C|c,X) \geq \bar{v}(\bar{p},\bar{C}|c,\bar{X})$.

The almost banal character of (61) is best revealed by an example from a more familiar situation. Let (A, B) be a dinner consisting of a main course A and a dessert B. If a dessert B is considered indifferent to $\bar{B}$ <u>after</u> a main course A, $(B|A) \sim (\bar{B}|A)$, then also the complete dinners should be regarded as indifferent, $(A,B) \sim (A,\bar{B})$, and inversely.

As is shown in Appendix 1 (61) implies that the ordinal indirect utility function of the consumer $v(p,C|c,X)$ and ordinal indirect utility function of the income user (60) expressed as function of C instead of Y are increasing transformations of each other:

(62) $v(p,C|c,X) = g(\bar{v}(p,C|c,X))$

$= g(\psi(p,C/c,c;\ X))$,

where $g: \mathbb{R} \to \mathbb{R}$ is an increasing function. By inserting

(63) $\partial v(p,C|c,X)/\partial p_i$

$= g'(\bar{v}(p,C|c,X))\partial\bar{v}(p,C|c,X)/\partial p_i$

$= g'(\psi(p,C/c,c;\ X))\partial\psi(p,C/c,c;\ X)/\partial p_i$

in (59) the demand functions $h^i(p,C|c,X)$ may be calculated from the indirect utility function $\psi(p,Y,c;\ X)$ of the income user in the way analogous to Roy's Theorem:

(64) $$h^i(p,C|c,X) = C\frac{\partial\psi(p,C/c,c;\ X)/\partial p_i}{\Sigma\ p_j\partial\psi(p,C/c,c;\ X)/\partial p_j} .$$

This is a straightforward generalization of Theorem 1 to cases where c differs from unity. To sum up we present the results in

<u>Theorem 2.</u> Suppose that the demand function $h(p,C|c,X)$ satisfies CUH with an implicite indirect utility function $v(p,C|c,X)$

satisfying (61). Then $v(p,C|c,X)$ and $\bar{v}(p,C|c,X) = \psi(p,C/c,c;\ X)$ are increasing transformations of each other both satisfying conditions $\bar{A}$ of an indirect utility function. Furthermore, the demand function $h(p,C|c,X)$ has the representation

$$(65)\qquad h(p,C|c,X) = C\frac{\nabla\psi(p,C/c,c;\ X)}{\nabla\psi(p,C/c,c;\ X)\cdot p} = C\frac{\nabla\bar{v}(p,C|c,X)}{\nabla\bar{v}(p,C|c,X)\cdot p} \quad \text{or}$$

$$(65b)\qquad h(r,1|c,X) = \frac{\nabla\psi(r,1/c,c;\ X)}{\nabla\psi(r,1/c,c;\ X)\cdot r} = \frac{\nabla\bar{v}(r,1|c,X)}{\nabla\bar{v}(r,1|c,X)\cdot r},\ r = p/C.$$

Theorem 2 provides the motivation for the following straightforward definition.

We call an indirect utility function $\psi(p,Y,c;\ X)$ <u>regular</u> if the function $\bar{v}(p,C|c,X) = \psi(p,C/c,c;\ X)$ of (p,C) has the properties $\bar{A}$ of an indirect utility function and is continuous in (p,C,c). The corresponding direct utility function having properties B in consumption space is determined by duality,

$$(66)\qquad u(q|c,X) = \min_{p}\{\bar{v}(p,C|c,X)\,|\,p\cdot q \le C\},$$

and the consumer demand function $h(p,C|c,X)$ is assumed to maximize $u(q|c,X)$ in the budget set. The direct utility function (66) may be independent of c (which is an assumption in ordinary demand theory) or they may differ for different c's. Anyhow, in the regular case $h(p,C|c,X)$ may be calculated straightforwardly from (65).

<u>Theorem 3.</u> A regular $\psi(p,Y,c;\ X)$ is necessarily also p-regular.

<u>Proof.</u> Let's consider $\psi(\lambda p^0,\lambda Y^0,c^0,X^1) = \psi(\lambda p^0,\lambda C^0/c^0,c^0,X^1) = \bar{v}(\lambda p^0,\lambda C^0|c^0,X^1)$. Because $\bar{v}(p,C|c,X)$ satisfies properties $\bar{A}$ it is also homogenous of degree zero in (p,C) so that $\bar{v}(\lambda p^0,\lambda C^0|c^0,X^1) = \bar{v}(p^0,C^0|c^0,X^0) = \psi(p^0,C^0/c^0,c^0;\ X^0) = \psi(p^0,Y^0,c^0;\ X^0)$, which proves the p-regularity. □

We have used the function $\psi(p,Y,c;\ X)$ to define the regular case. It is worth while writing the definition of regularity also in terms of $V(p,C,S;\ X)$. By p-regularity we have $V(p,C,S;\ X) = V(p/C,1,\frac{S}{C};\ X) = V(r,1,\frac{1-c}{c};\ X)$, where $r = p/C$. Therefore we have the theorem

<u>Theorem 4.</u> The utility function $\psi(p,Y,c;\ X)$ is regular if and only if the function $V(r,1,\frac{c}{1-c};\ X)$ of $r = p/C$ has the properties A of an indirect utility function for any $c = C/Y$ and is continuous in c.

<u>Proof.</u> Regularity of $\psi(p,Y,c;\ X)$ means that $\bar{v}(p,Y|c,X) = \psi(p,Y,c;\ X)$ fulfills conditions $\bar{A}$ as a function of (p,Y) for any c and is continuous in c. Equivalently, the function $\tilde{v}(r|c,X) = \bar{v}(p/C,Y/C|c,X) = \bar{v}(r,\frac{1}{c}|c,X) = \psi(r,\frac{1}{c},c;\ X)$ of $r = p/C$ must fulfill conditions A for any c and be continuous in c. But by equation (49) $\tilde{v}(r|c,X) = \psi(r,\frac{1}{c},c;\ X) = \psi(p,Y,c;\ X) = \psi(p,C+S,\frac{C}{C+S};\ X) = V(p,C,S;\ X) = V(r,1,\frac{c}{1-c};\ X)$, so that $\tilde{v}(r|c,X) = V(r,1,\frac{c}{1-c};\ X)$ and the theorem is proved. □

It is not true that in the regular case the function $V(p,C,S;\ X)$ is an indirect utility function in (p,C) for a fixed S. It is not natural to keep the money value of saving S as fixed when p and C change. In the definition of regularity the propensity to consume $c = C/Y$ (or equivalently the propensity to save $S/Y = 1-c$) was kept constant.

Regularity of $\psi(p,Y,c;\ X)$ is a kind of separability assumption, which makes it possible to allocate the disposable income into saving and different consumption categories in two stages:

<u>First:</u> Y is allocated into $\hat{C} = \hat{c}Y$ and $\hat{S} = (1-\hat{c})Y$ by setting the marginal utility $\partial\psi(p,Y,c;\ X)/\partial c$ of c equal to zero and solving for the optimal propensity to consume $\hat{c}$.

<u>Second:</u> $\hat{C}$ is allocated into consumption categories according to the indirect utility function $\bar{v}(p,\hat{C}|\hat{c},X) = \psi(p,\hat{C}/\hat{c},\hat{c};\ X)$.

In the second stage the allocation may be carried out simply using Roy's Theorem

$$(67) \qquad h(p,\hat{C}|\hat{c};\ X) = \hat{C}\,\frac{\nabla\bar{v}(p,\hat{C}|\hat{c},X)}{\nabla\bar{v}(p,\hat{C}|\hat{c},X)\cdot p}$$

$$= \hat{C}\,\frac{\nabla\psi(p,\hat{C}/\hat{c},\hat{c};\ X)}{\nabla\psi(p,\hat{C}/\hat{c},\hat{c};\ X)\cdot p},$$

where $\hat{q} = h(p,\hat{C}|\hat{c},X)$ is the optimal consumption bundle corresponding to the optimal consumption expenditure-saving pair $(\hat{C},\hat{S})$. The demand system $h(p,\hat{C}|\hat{c},X)$ is continuous in $(p,\hat{C},\hat{c})$. If $\hat{C} = \hat{C}(p,Y;\ X)$ and $\hat{S} = \hat{S}(p,Y;\ X)$ are continuous in (p,Y) then $h(p,\hat{C}|\hat{c},X) = h(p,\hat{C}(p,Y;\ X)|\hat{C}(p,Y;\ X)/Y;\ X) = h^*(p,Y;\ X)$ is also continuous in p and Y. Of course the demand (67) also maximizes $u(q|\hat{c},X)$ under the budget constraint $p\cdot q \leq \hat{C}$.

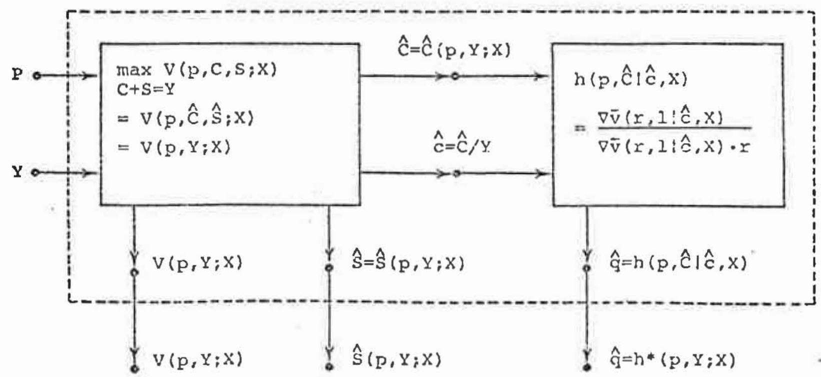
Deaton (1977, p. 901) proposes a similar two stage procedure but his consumption expenditure function and demand functions were not derived from any utility considerations.

The two stage procedure of optimization is perhaps understood more clearly through the <u>total indirect utility of income</u> Y in the face of prices p. It is defined by

$$(68) \qquad V(p,Y;\ X) = V(p,\hat{C}(p,Y;\ X),\ \hat{S}(p,Y;\ X);\ X),$$

where the allocation of Y into the optimal consumption expenditure $\hat{C}(p,Y;\ X)$ and saving $\hat{S}(p,Y;\ X)$ is taken into account. In $V(p,C,S;\ X)$ variables C and S were "freely varying" and not optimized as in (68). The procedure is summarized in figure 9.

<u>Figure 9:</u> The two stage procedure of determining the optimal consumption expenditure-saving pair $(\hat{C},\hat{S})$ and the optimal consumption bundle $\hat{q} = h(p,\hat{C}|\hat{c},X) = h^*(p,Y;\ X)$ from given prices and income



Now we are able to see more clearly also the difference between the (more or less) indirect utility function $V(p,C,S;\ X)$ and (more or less) direct utility function $U(q,p,S;\ X)$ characterized by (28). In fact we have

$$(69)\qquad V(p,Y;\ X) = V(p,\hat{C},\hat{S};\ X) = U(\hat{q},p,\hat{S};\ X)\ ,$$

where optimal choices of C,S and q are inserted. Or in other notation

$$(70)\qquad \max_{(q,S)}\{U(q,p,S;\ X)\,|\,p\cdot q + S = Y\}$$

$$= \max_{C+S=Y} V(p,C,S;\ X) = V(p,Y;\ X)\ .$$

The most natural starting point to analyze the decisions of a consumer-saver seems to be offered by $V(p,C,S;\ X)$, while other representations of the same preferences are more difficult to start with.

In Appendix 2 some examples of the regular case are given

6. COMPENSATED INCOME IN CONSUMPTION-SAVING ECONOMY WHEN PRICES ARE CHANGED

Now we are equipped strongly enough to return to our original problem. Let's consider a basic situation (p^0,Y^0) determined by 'old' prices and disposable income. For notational convenience we will now drop other relevant variables X from our functions.

The optimal (C,S) pair is

$$(71)\qquad C^0 = \hat{C}(p^0,Y^0),\ S^0 = \hat{S}(p^0,Y^0)$$

with $C^0 + S^0 = Y^0$. The pair (C^0,S^0)maximizes $V(p^0,C,S)$ when $C + S = Y^0$ the maximum utility being

$$(72)\qquad V^0 = V(p^0,C^0,S^0) = V(p^0,Y^0).$$

Because (72) is the maximum of $V(p^0,C,S)$ when $C + S = Y^0$ the utility of consuming all income, i.e. the utility of the choice, $C = Y^0$ and $S = 0$ is smaller than $V(p^0,C^0,S^0)$:

$$(73)\qquad V(p^0,Y^0,0) \leq V(p^0,C^0,S^0) = V^0.$$

Equality holds only if $S^0 = 0$ by happy accident or the indifference surface $V(p^0,C,S) = V^0$ coincides with the income constraint $Y^0 = C + S$. The utility $V(p^0,Y^0,0)$ in (73) corresponds to the situation where saving is forced to zero and no compensation of this is given to the consumer-saver.

Essentially (73) is a revealed preference argument: the choice (C^0,S^0) is revealed better than $(Y^0,0)$, which was also possible but was not chosen.

But we may also ask what would be the income $\bar{Y}^0$ for which the utility $V(p^0,\bar{Y}^0,0)$ of consuming $\bar{Y}^0$ and saving nothing, $S = 0$, is equal to old utility V^0:

(74) $$V(p^0,\bar{Y}^0,0) = V^0.$$

Generally the solution $\bar{Y}$ of

(75) $$V(p,\bar{Y},0) = V$$

defines a function $\bar{Y} = \bar{Y}(p,V)$ of prices and utility; $\bar{Y}(p,V)$ is the minimum income for which consuming $\bar{Y}(p,V)$ and saving nothing would produce a given utility V.

Therefore $V(p,\bar{Y}(p,V),0) = V$ identically for all admissible p and V. The function $\bar{Y}(p,V)$ is a generalization of the cost or expenditure function $C(p,u)$ defined by (1) in all-consumption economies. We may alternatively define $\bar{Y}(p,V)$ by

(76) $$\bar{Y}(p,V) = \min_{\bar{Y}}\{\bar{Y} \mid V(p,\bar{Y},0) \geq V\}.$$

Another generalization of $C(p,u)$ in consumption-saving economy is defined as follows. Let $\hat{C}(p,Y)$ and $\hat{S}(p,Y)$ be the optimal consumption and saving pair in the face of (p,Y) and determine the income Y such that

(77) $$V(p,\hat{C}(p,Y),\hat{S}(p,Y)) = V(p,Y) = V.$$

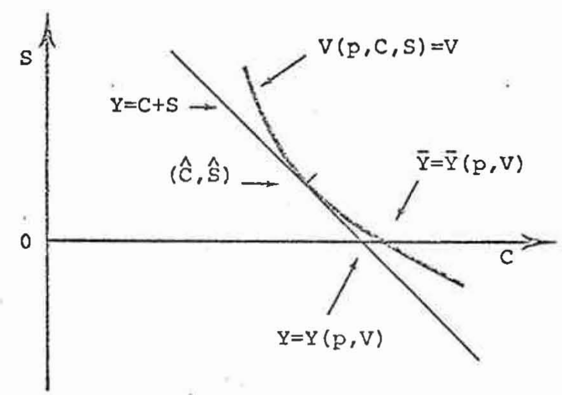
The solution $Y = Y(p,V)$ depends on prices and the total utility level and $Y(p,V)$ is the minimum income which quarantees a utility level V under prices p. Alternatively it may be defined by

(78) $$Y(p,V) = \min_{Y}\{Y \mid V(p,Y) \geq V\}$$

$$= \min_{(C,S)}\{Y \mid Y = C + S \ \& \ V(p,C,S) \geq V\}.$$

Figure 10 illustrates the definitions.

Figure 10. Definitions of $Y(p,V)$ and $\bar{Y}(p,V)$



It is clear that $Y(p,V)$ cannot exceed $\bar{Y}(p,V)$, i.e.

$$Y(p,V) \leq \bar{Y}(p,V) \tag{79}$$

for all admissible (p,V)-pairs.

The difference $\bar{Y} - Y = \bar{Y}(p,V) - Y(p,V) \geq 0$ is the money value of the compensation (in dollars, say) if the saving is forced to zero.

Similarly the ratio

$$\frac{\bar{Y}}{Y} = \frac{\bar{Y}(p,V)}{Y(p,V)} \geq 1 \tag{80}$$

is the relative compensation needed to retain the utility level V under prices p if the saving is forced to zero.

Denote by

$$C(p,V) = \hat{C}(p,Y(p,V)) \tag{81}$$

$$S(p,V) = \hat{S}(p,Y(p,V)) \tag{82}$$

the compensated consumption and saving functions, respectively. The pair $(C(p,V),S(p,V))$ is the optimal consumption-saving allocation producing the utility V under prices p, so that we have for all admissible p and V

$$C(p,V) + S(p,V) = Y(p,V) \tag{83}$$

$$V(p,C(p,V),S(p,V)) = V(p,Y(p,V)) = V. \tag{84}$$

Note also that

$$V(p,Y(p,V),0) \leq V(p,C(p,V),S(p,V)) = V(p,\bar{Y}(p,V),0) = V. \tag{85}$$

Consider now a change (say an increase) in prices, $p^0 \to p^1$. If prices increase, more income is needed to retain the old total utility level $V^0 = V(p^0,C^0,S^0)$, where $C^0 = \hat{C}(p^0,Y^0) = C(p^0,V^0)$ and $S^0 = \hat{S}(p^0,Y^0) = S(p^0,V^0)$.

What is the income needed in the new price situation p^1 to attain the old utility level? In the old situation $Y(p^0,V^0) = C(p^0,V^0) + S(p^0,V^0)$ was just enough to quarantee the utility level V^0. In the face of new prices the same utility V^0 is just reached if income were

$$Y(p^1,V^0) = C(p^1,V^0) + S(p^1,V^0). \tag{86}$$

Here $(C(p^1,V^0),S(p^1,V^0))$ is the optimal consumption-saving allocation which produces the old utility V^0 under new prices p^1 and $Y(p^1,V^0)$ is the compensated income, where the price change $p^0 \to p^1$ is compensated.

The difference $Y(p^1,V^0) - Y(p^0,V^0) = Y(p^1,V^0) - Y^0$ tells how much more income is needed to retain the old utility level when prices have changed, $p^0 \to p^1$. Therefore $Y(p^1,V^0) - Y(p^0,V^0)$ is the money value of the compensation needed after a price change. The relative compensation

$$(87) \qquad P(p^1,p^0;V^0) = \frac{Y(p^1,V^0)}{Y(p^0,V^0)} = \frac{Y(p^1,V^0)}{Y^0} = \frac{C(p^1,V^0) + S(p^1,V^0)}{C^0 + S^0}$$

is a natural generalization of the (Laspeyres' type of) Konüs cost of living index (4) in consumption-saving economies. Shortly: $P(p^1,p^0;V^0)$ is the relative cost of old utility $V^0 = V(p^0,C^0,S^0)$ in two price situations. We will refer to $P(p^1,p^0;V^0)$ also as the price index of the disposable income.

The formal analogy of (87) with the corresponding definition in all-consumption economies may be also presented as follows. The compensated income $Y(p^1,V^0) = \tilde{Y}^1$ leads to the same utility under p^1-prices as Y^0 leads under p^0-prices:

$$(88) \qquad V(p^1,\tilde{Y}^1) = V(p^0,Y^0) = V^0.$$

The implied true cost of living index is $\tilde{Y}^1/Y^0 = Y(p^1,V^0)/Y(p^0,V^0)$. The procedure is illustrated in figure 11.

Figure 11. Determination of the relative cost of the old utility level in two price situations p^0 and p^1 as measured by $P(p^1,p^0;V^0) = \tilde{Y}^1/Y^0$.

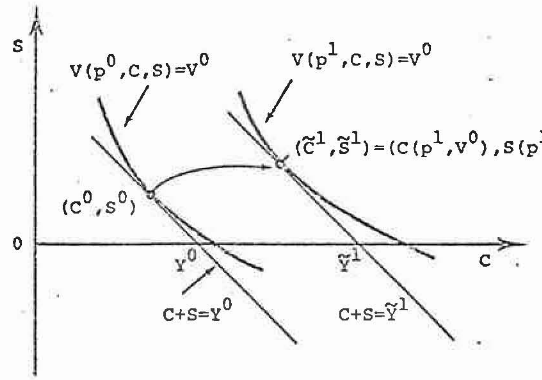


Figure 11 corresponds to a substantial increase in prices. The indifference curve $\{(C,S)|V(p^1,C,S) = V^0\}$ consists of those (C,S)-pairs which would give the same satisfaction V^0 under p^1-prices as the optimal (C^0,S^0)-pair gave under p^0-prices. The pair $(\tilde{C}^1,\tilde{S}^1) = (C(p^1,V^0),S(p^1,V^0))$ is the cheapest consumption-saving combination producing old utility V^0 under p^1-prices so that the needed compensated income is $\tilde{C}^1 + \tilde{S}^1 = C(p^1,V^0) + S(p^1,V^0) = Y(p^1,V^0) = \tilde{Y}^1$. If the consumer-saver would get income $\tilde{Y}^1$ under p^1-prices he would be just able to retain his old utility level V^0. Note that because both consumption and saving adjust to the new price situation the propensity to consume may change somewhat: C^0/Y^0 and $\tilde{C}^1/\tilde{Y}^1$ need not be equal to each other.

Another generalization of the Konüs cost of living index $P(p^1,p^0;u) = C(p^1,u)/C(p^0,u)$ uses the generalized cost function $\bar{Y}(p,V) = \min_{\bar{Y}}\{\bar{Y}|V(p,\bar{Y},0) \geq V\}$ of (76), where saving is forced to zero. $\bar{Y}(p^0,V^0) = \bar{Y}^0$ is the minimum income for which $(\bar{Y}^0,0)$ gives the same total utility V^0 as (C^0,S^0). Similarly $Y(p^1,V^0) = \bar{Y}^1$ is the income for which $(\bar{Y}^1,0)$, i.e. consuming $\bar{Y}^1$ and saving nothing, gives the same total utility V^0 under new prices p^1; $\bar{Y}^1$ is a special kind of compensated income which quarantees the old utility after a price change $p^0 \to p^1$. The relative compensation

$$(89) \qquad \bar{P}(p^1,p^0;V^0) = \frac{\bar{Y}(p^1,V^0)}{\bar{Y}(p^0,V^0)} = \frac{\bar{Y}^1}{\bar{Y}^0}$$

is another generalization of the Konüs cost of living index. In fact it <u>is formally</u> a Konüs cost of living index because both $\bar{Y}^1$ and $\bar{Y}^0$ are actually used totally in purchases of consumer goods and therefore the calculation of $\bar{P}(p^1,p^0;V^0)$ is carried out in consumption space by Konüs principle. When saving is exogenized to zero, $S = 0$ or $c = 1$, the total utility function $V(p,C,0) = \psi(p,C,1)$ has the properties $\bar{A}$ of an indirect utility function by weak CUH. Therefore there exists by duality also a direct utility function $u(q|1)$ representing the same consumer preferencies in the quantity space and the generalized cost function $\bar{Y}(p,V)$ is an ordinary cost function (1) corresponding to these preferencies:

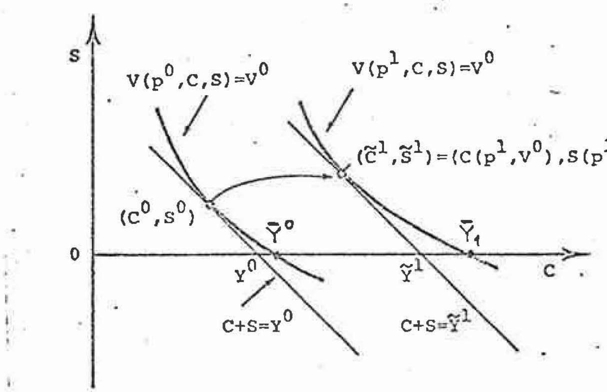
$$(90) \qquad \bar{Y}(p,V) = \min\{C|C \geq p\cdot\bar{q} \ \& \ u(\bar{q}|1) = V\}$$

Also

$$(91) \qquad \bar{P}(p^1,p^0;\ V^0) = \frac{p^1\cdot h(p^1,\bar{Y}^1|1)}{p^0\cdot h(p^0,\bar{Y}^0|1)},$$

where $h(p,Y|1)$ optimizes $u(q|1)$ under the budget constraint $p\cdot q \leq Y$. The calculation of (89) is illustrated in figure 12 where only incomes $\bar{Y}^0$ and $\bar{Y}^1$ are added to figure 11.

<u>Figure 12.</u> Determination of the relative cost of the old utility level in two price situations as measured by $\bar{P}(p^1,p^0;\ V^0) = \bar{Y}^1/\bar{Y}^0$.



Here $\bar{Y}^0$ and $\bar{Y}^1$ are the points where the 'old' and 'new' indifference curves intersect the C-axis. Therefore using the incomes $\bar{Y}^0$ and $\bar{Y}^1$ totally in purchases of consumption goods under prices p^0 and p^1 respectively would just give the old utility level: $V^0 = V(p^0,C^0,S^0) = V(p^0,\bar{Y}^0,0) = V(p^1,\bar{Y}^1,0)$. It is clear from the figure that usually $\tilde{Y}^1/Y^0 \approx \bar{Y}^1/\bar{Y}^0$ so that $P(p^1,p^0;\ V^0)$ are $\bar{P}(p^1,p^0;\ V^0)$ are good approximations of each other.

7. ESTIMATION OF THE COMPENSATED INCOME

The hypothetical incomes $\bar{Y}^0$ and $\bar{Y}^1$ may be both regarded as auxiliary characteristics, which are not very interesting in themselves. The main aim of our analysis is to determine the compensated income

$$\text{(92)} \qquad \tilde{Y}^1 = Y(p^1,V^0) = C(p^1,V^0) + S(p^1,V^0)$$

the use of which for consumption and saving would just give the old utility in the new price situation. Alternatively if the generalized price index or relative compensation (87) is known then $\tilde{Y}^1$ may be determined from

$$\text{(93)} \qquad \tilde{Y}^1 = Y(p^1,V^0) = P(p^1,p^0;\ V^0)Y^0 .$$

Because $P(p^1,p^0;\ V^0) \approx \bar{P}(p^1,p^0;\ V^0) = \bar{Y}^1/\bar{Y}^0$ we have also

$$\text{(94)} \qquad \tilde{Y}^1 \approx \bar{P}(p^1,p^0;\ V^0)Y^0 .$$

so that $\bar{P}(p^1,p^0;\ V^0)Y^0 = (\bar{Y}^1/\bar{Y}^0)Y^0$ is as a rule a very good estimate of $\tilde{Y}^1$. The approximation (94) is very good when the relative compensations $\bar{Y}^0/Y^0 \geq 1$ and $\bar{Y}^1/\tilde{Y}^1 \geq 1$ of forcing the saving to zero are approximately equal. This is shown as follows

$$\text{(95)} \qquad \bar{P}(p^1,p^0;\ V^0) = \frac{\bar{Y}^1}{\bar{Y}^0} \qquad \text{(definition)}$$

$$= \frac{\tilde{Y}^1}{Y^0}\ \frac{\bar{Y}^1/\tilde{Y}^1}{\bar{Y}^0/Y^0} \qquad \text{(identity)}$$

$$\approx \frac{\tilde{Y}}{Y^0} \qquad (\text{if } \bar{Y}^1/\tilde{Y}^1 \approx \bar{Y}^0/Y^0)$$

$$= P(p^1,p^0;V^0) . \qquad \text{(definition)}$$

We usually have $\bar{Y}^0/Y^0 \approx \bar{Y}^1/\tilde{Y}^1$ even if $\bar{Y}^0$ and $\bar{Y}^1$ are considerably greater than Y^0 and $\tilde{Y}^1$ and e.g. $\bar{Y}^1$ is a bad approximation of $\tilde{Y}^1$.

The result (92) is conceptually important although it is mathematically rather trivial. Because of (94) it is sufficient to derive a good approximation of $\bar{P}(p^1,p^0;\ V^0)$ instead of $P(p^1,p^0;\ V^0)$. The former $\bar{P}(p^1,p^0;\ V^0) = \bar{Y}^1/\bar{Y}^0$ is easier to estimate because here saving is forced to zero and the calculations occur completely in the consumption space.
On the other hand, in estimating the price index of the disposable income

$$\text{(96)} \qquad P(p^1,p^0;\ V^0) = \frac{\tilde{Y}^1}{Y^0} = \frac{C(p^1,V^0) + S(p^1,V^0)}{C^0 + S^0}$$

directly we should in principle discuss how both compensated consumption and saving react when prices are changed. We may so to speak eliminate saving in the first phase by forcing it to zero (by consuming the actual saving) so that all income is consumed.

This means that saving is considered from the point of view of its alternative use, consuming, and income is considered as if it were all consumed.

The discussion is complicated by allowing for the relative compensations $\bar{Y}^0/Y^0$ and $Y^1/\tilde{Y}^1$ (see (95) and figure 12) needed to eliminate the welfare effects of forcing saving to zero. We have

$$(97)\qquad \frac{\bar{Y}^0}{Y^0} = \frac{\bar{Y}(p^0,V^0)}{Y(p^0,V^0)}$$

$$(98)\qquad \frac{Y^1}{\tilde{Y}^1} = \frac{\bar{Y}(p^1,V^0)}{Y(p^1,V^0)}$$

so that both are ratios

$$(99)\qquad \frac{\bar{Y}(p,V^0)}{Y(p,V^0)}$$

between the two generalized minimum cost functions calculated at the same (p,V^0)-pair, see figure 10. The ratio (99) is always at least 1 by (80) and it depends only on curvature properties of the indifference curve $\{(C,S)|V(p,C,S) = V^0\}$. We shortly sketch a general representation for the ratio (99). This allows us to estimate the accuracy of (94) and (95) and improve the accuracy if it is considered worth while doing so. Usually it is unnecessary.

Represent the indifference curve $A(p,V) = \{(C,S)|V(p,C,S) = V\}$ for given (p,V) (see figure 10) as a function of S, $C = f(S)$, and expand it to a Taylor series with remainder at $\hat{S} = S(p,V)$:

$$(100)\qquad \begin{aligned} f(S) &= f(\hat{S}) + f'(\hat{S})(S-\hat{S}) + \tfrac{1}{2}f''(\bar{S})(S-\hat{S})^2 \\ &= \hat{C} - 1(S-\hat{S}) + \tfrac{1}{2}f''(\bar{S})(S-\hat{S})^2 \\ &= (\hat{C}+\hat{S}) - S + \tfrac{1}{2}f''(\bar{S})(S-\hat{S})^2 , \end{aligned}$$

where $\bar{S}$ is some point between S and $\hat{S}$. We have $f(0) = \bar{Y} = \bar{Y}(p,V)$, $\hat{C} + \hat{S} = Y(p,V)$ and therefore

$$(101)\qquad f(0) = \bar{Y}(p,V) = Y(p,V) + \tfrac{1}{2}f''(\bar{S})\hat{S}^2 ,$$

where $\bar{S}$ now lies between 0 and $\hat{S}$.

Denoting half of the mean curvature $\frac{1}{2}f''(\bar{S})$ of the indifference curve $A(p,V)$ along the interval $(\min(0,\hat{S}), \max(0,\hat{S}))$ by $a(p,V)$ we have by (99)

$$(102)\qquad \begin{aligned} a(p,V) &= \tfrac{1}{2}f''(\bar{S}) \\ &= \frac{\bar{Y}(p,V) - Y(p,V)}{\hat{S}^2} , \end{aligned}$$

which shows that usually $a(p,V) > 0$ and $a(p,V) \geq 0$ always because $\bar{Y}(p,V) \geq Y(p,V)$ and $\hat{S}^2 \geq 0$. We call $a(p,V)$ the curvature parameter. It follows that

(103) $$\frac{\tilde{Y}(p,V)}{Y(p,V)} = \frac{Y(p,V) + a(p,V)\hat{S}^2}{Y(p,V)}$$

$$= 1 + a(p,V)\frac{S(p,V)}{Y(p,V)} S(p,V)$$

$$= 1 + a(p,V)\, s(p,V)\, S(p,V)$$

so that the ratio $\tilde{Y}(p,V)/Y(p,V) \geq 1$ is a simple function of the curvature parameter $a(p,V)$, the propensity to save $s(p,V) = S(p,V)/Y(p,V) = \hat{S}/\hat{Y}$ and the optimal value of saving $\hat{S} = S(p,V)$, all determined for a given indifference curve $A(p,V)$.

Applying (103) for (97) and (98) we get

(104) $$\frac{\tilde{Y}^0}{Y^0} = \frac{\tilde{Y}(p^0,V^0)}{Y(p^0,V^0)}$$

$$= 1 + a(p^0,V^0)\, s\,(p^0.V^0)\, S\,(p^0,V^0)$$

$$= 1 + a^0 s^0 S^0$$

(105) $$\frac{\bar{Y}^1}{\tilde{\bar{Y}}^1} = \frac{\tilde{Y}(p^1,V^0)}{Y(p^1,V^0)}$$

$$= 1 + a(p^1,V^0)\, s\,(p^1,V^0)\, S\,(p^1,V^0)$$

$$= 1 + \bar{a}^1\bar{s}^1\bar{S}^1 ,$$

where the bars (-) in $\bar{a}^1\bar{s}^1\bar{S}^1$ indicate that e.g. $\bar{S}^1 = S(p^1,V^0)$ usually differs from actual saving $S^1 = S(p^1,V^1)$ in the new situation (p^1,V^1). For the ratio of (105) and (104) we get

(106) $$\frac{\bar{Y}^1/\tilde{\bar{Y}}}{\tilde{Y}^0/Y^0} = \frac{1 + \bar{a}^1\bar{s}^1\bar{S}^1}{1 + a^0 s^0 S^0}$$

$$\approx 1 + (\bar{a}^1\bar{s}^1\bar{S}^1 - a^0 s^0 S^0)$$

In order to improve the approximation (95) in estimating $P(p^1,p^0;\, V^0)$ we start from the identity

(107) $$P(p^1,p^0;\, V^0) = \bar{P}(p^1,p^0;\, V^0)\frac{\tilde{Y}^0/Y^0}{\bar{Y}^1/\tilde{\bar{Y}}^0}$$

$$= \bar{P}(p^1,p^0;\, V^0)\frac{1 + a^0 s^0 S^0}{1 + \bar{a}^1\bar{s}^1\bar{S}^1} ,$$

where the left hand side characteristics are first estimated in an actual situation. The price index $\bar{P}(p^1,p^0;\, V^0)$ is defined completely in consumption space and it is estimated by standard methods of demand theory. The consumer price index calculated by Laspeyres formula provides as a rule a sufficiently good approximation for it and more advanced methods are needed only if price relatives p_i^1/p_i^0 of consumer goods differ from each other considerably. In that case some superlative index number formula, e.g. Fisher, Törnqvist or Sato-Vartia formula should be used, see Vartia (1978). These formulas are calculated straightforwadly from <u>actual</u> consumption data and they provide an excellent approximation not directly for $\bar{P}(p^1,p^0;\ V^0)$ but for $\bar{P}(p^1,p^0;\ \bar{V})$, where $\bar{V}$ is an average consumption utility lying somewhere between $\bar{V}^0 = V(p^0,C^0,0)$ and $\bar{V}^1 = V(p^1,C^1,0)$. But unless the nonhomotheticity of the consumer utility function $u(q|1)$ or $v(p,C|1)$ is exceptionally strong while $\bar{V}$ and V^0 differ considerably $\bar{P}(p^1,p^0;\ \bar{V})$ and $\bar{P}(p^1,p^0;\ V^0)$ are almost equal to each other and any superlative consumer price index formula approximates well also $\bar{P}(p^1,p^0;\ V^0)$, cf. chapter 2 and Vartia (1976, p. 39), Theil (1967) and Diewert (1976). For any homothetic consumer utility $u(q|1)$ $\bar{P}(p^1,p^0;\ \bar{V})$ and $\bar{P}(p^1,p^0;\ V^0)$ are necessarily equal for all reference utility levels $\bar{V}$ and V^0. Therefore for practical purposes $\bar{P}(p^1,p^0;\ V^0)$ of (107) may be approximated well enough. The problem is solved when $a^0s^0S^0$ and $\bar{a}^1\bar{s}^1\bar{S}^1$ are estimated with a comparable accuracy. We illustrate their estimation by a few examples.

Example 1. If the preferencies of the income user are strongly p-regular (see Appendix 2) the indifference curves $A(p^0,V^0)$ and $A(p^1,V^0)$ corresponding to the same utility V^0 and two different price situations are 'homothetic', i.e. notwithstanding a scale factor λ their shape is the same. In this case we have: For all p^0 and p^1 there exists a λ such that for all (C,S) and V^0

(108) $$(C,S) \in A(p^0,V^0) \Rightarrow (\lambda C,\lambda S) \in A(p^1,V^0)\ .$$

Here $\lambda = P(p^1,p^0;\ V^0)$ which in turn equals $\bar{P}(p^1,p^0;\ V^0)$. Optimal propensities to save s^0 and $\bar{s}^1$ are also equal and $\bar{S}^1 = \lambda S^0 = P(p^1,p^0;\ V^0)S^0$. Because here $a^0s^0S^0 = \bar{a}^1\bar{s}^1\bar{S}^1$ we must have $\bar{a}^1 = a^0S^0/\bar{S}^1 = a^0/\lambda = a^0/\bar{P}(p^1,p^0;\ V^0)$. Thus the curvature parameter diminishes somewhat if prices increase, $\lambda > 1$. This relation $\bar{a}^1 = a^0/\bar{P}(p^1,p^0;\ V^0)$ is a reasonable approximation also more generally.

Example 2. If all prices change proportionally, $p^1 = \lambda p^0$ then by the p-regularity assumption (52) we have

(109) $$V(\lambda p^0,\lambda C^0,\lambda S^0) = V(p^0,C^0,S^0)$$

for all (p^0,C^0,S^0) and $\lambda > 0$. In other notation: For all p^0, $\lambda > 0$, (C^0,S^0) and V^0

(110) $$(C^0,S^0) \in A(p^0,V^0) \Rightarrow (\lambda C^0,\lambda S^0) \in A(p^1,V^0)\ .$$

Therefore for all proportional changes in prices we have

(110) $$P(\lambda p^0,p^0;\ V^0) = \bar{P}(\lambda p^0,p^0;\ V^0) = \lambda\ .$$

Also here $\bar{s}^1 = s^0$, $\bar{S}^1 = \lambda S^0$ and $\bar{a}^1 = a^0/\lambda$. Equation (110) holds approximately if prices change almost proportionally.

Example 3. Suppose that p^0 and p^1 are not proportional to each other and that the relative prices p_i^1/p_i^0 of consumption goods have changed e.g. in such a way that saving has become less desirable, $\bar{s}^1 < s^0$. The compensated value of saving $\bar{S}^1 = S(p^1,V^0)$ may be estimated as follows

$$(111) \qquad \bar{S}^1 = S(p^1,V^0)$$

$$= s(p^1,V^0)Y(p^1,V^0)$$

$$= \bar{s}^1\tilde{Y}^1$$

$$= \bar{s}^1 \frac{\tilde{Y}^1}{Y^0} Y^0$$

$$= \bar{s}^1 P(p^1,p^0;\ V^0)Y^0$$

$$\approx \bar{s}^1 \bar{P}(p^1,p^0;\ V^0)Y^0 .$$

For the curvature parameter we use the approximation

$$(112) \qquad \bar{a}^1 \approx a^0/\bar{P}(p^1,p^0;\ V^0) .$$

This means roughly that the curvatures of the two indifference curves drawn on <u>real</u> (C,S)-plane are approximately equal near the optimum point. This hold e.g. if the indifference curve has shifted on the real (C,S)-plane. Therefore

$$(113) \qquad \bar{a}^1\bar{s}^1\bar{S}^1 \approx a^0(\bar{s}^1)^2Y^0$$

and

$$(114) \qquad \frac{1 + a^0s^0S^0}{1 + \bar{a}^1\bar{s}^1\bar{S}^1} \approx \frac{1 + a^0(s^0)^2Y^0}{1 + a^0(\bar{s}^1)^2Y^0}$$

$$\approx 1 - a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0$$

Here the old income Y^0 and the propensity to save s^0 are known and only a^0 and $\bar{s}^1$ need to be estimated.

Their values must be inferred from the details of the situation investigated.

Suppose that Y^0 = 50 000 mk/year, s^0 = 0.20 and thus (C^0,S^0) = (40 000, 10 000). If the relative compensation

$$(115) \qquad \frac{\bar{Y}^0}{Y^0} = 1 + a^0s^0S^0$$

of forcing saving S^0 = 10 000 to zero equals 1.02 the pair (C,S) = (52 000,0) would give the same satisfaction to our income user as (40 000,10 000) actually did. By an extra 2 000 mk (which is 20 % of his saving) he could be persuaded to consume all his income. Here the curvature parameter a^0 would be thus

$$(116) \qquad a^0 = (\frac{\bar{Y}^0}{Y^0} - 1)/s^0S^0$$

$$= 0.02/(0.20)(10000)$$

$$= 0.000010 .$$

Note that if our income user would be almost indifferent between saving or consuming his actual saving S^0, the compensation $\bar{Y}^0/Y^0 \geq 1$ would be almost unity and a^0 very nearly zero. In this case (114) would be practically unity and the the indifference surfaces $\Lambda(p^0,V^0)$ and $\Lambda(p^1,V^0)$ would almost coincide with the budget constraints $Y^0 = C + S$ and $\tilde{Y}^1 = C + S$ respectively. In that case $P(p^1,p^0;\ V^0) =$

$\bar{P}(p^1, p^0; V^0)$ would again be an excellent approximation and our theory would reduce to the traditional approach discussed in chapter 2.

But here we consider the more interesting case where the curvatures of the indifference curves $A(p^0, V^0)$ and $A(p^1, V^0)$ differ from zero, so that an estimate of the new compensated propensity to save $\bar{s}^1$ is needed. Suppose that $\bar{P}(p^1, p^0; V^0) =$ 1.046 and that the subjective profitability of saving has decreased because of nonequal relative prices p_i^1/p_i^0 in such a way that $\bar{s}^1$ is estimated to be 0.18 instead of previous $s^0 = 0.20$. In this case we get from (114)

$$(117) \qquad 1 - a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0$$

$$= 1 - 0.00001(0.18 - 0.20)(0.18 + 0.20)50\,000$$

$$= 1.0038$$

By inserting this and our estimate 1.046 of the consumer price index $\bar{P}(p^1, p^0; V^0)$ to

$$(118) \qquad P(p^1, p^0; V^0) \approx \bar{P}(p^1, p^0; V^0)(1 - a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0)$$

we get finally $P(p^1, p^0; V^0) = (1.046)(1.0038) = 1.05$, which falls short of the consumer price index $\bar{P}(p^1, p^0; V^0) = 1.046$ by less than half a percent. Therefore even when the propensity to save changed from $s^0 = 0.20$ to $\bar{s}^1 = 0.18$ the price index $\bar{P}(p^1, p^0; V^0)$ remained an good approximation of the price index $P(p^1, p^0; V^0)$ of the disposable income.

Note also that (118) reduces to our basic approximation $P(p^1, p^0; V^0) \approx \bar{P}(p^1, p^0; V^0)$ when $\bar{s}^1$ and s^0 are estimated to be equal.

The compensated income in the new price situation p^1 is calculated from (93)

$$(118) \qquad \tilde{Y}^1 = Y(p^1, V^0)$$

$$= P(p^1, p^0; V^0)Y^0$$

$$\approx (1.05)50\,000$$

$$= 52\,500$$

and the optimal saving and consumption are determined as follows

$$(119) \qquad S(p^1, V^0) = s(p^1, V^0)Y(p^1, V^0)$$

$$= \bar{s}^1\tilde{Y}^1$$

$$\approx (0.18)52\,500$$

$$= 9\,450 \text{ (mk/year)}$$

$$(120) \qquad C(p^1, V^0) = Y(p^1, V^0) - S(p^1, V^0)$$

$$= \tilde{Y}^1 - \bar{S}^1$$

$$\approx 52\,500 - 9\,450$$

$$= 43\,050 \text{ (mk/year)}.$$

Thus the compensated income $Y(p^1,V^0) = 52\ 500$ mk/year is divided optimally into compensated consumption $C(p^1,V^0) =$ 43 050 mk/year and compensated saving 9 450 mk/year, which are just sufficient to produce the old utility $V^0 = V(p^0,C^0,S^0) = V(p^0, 40\ 000, 10\ 000)$ under new prices p^1. The rather great relative increase in consumption $\frac{C(p^1,V^0)}{C^0} - 1 = 43\ 050/40\ 000 - 1 = 0.0763 = 7.63$ % (which exceeds 5 %) is offset by the relative decrease of $\frac{S(p^1,V^0}{S^0} - 1 = \frac{9\ 450}{10\ 000} - 1 = -0.0550 = -5.50$ % in saving. Note that it is by no means sufficient to give a 4.60 % (or even 5 %) compensation for consumption only (i.e. give for consumption purposes (1.0460)(40 000) = 41 840 mk/year) and leave saving uncompensated. In this case the income in the new price situation would be 41 840 + 10 000 = 51 840 mk/year which exceeds the old income 50 000 by only 3.68 %. The truly compensated income (118) must exceed the old income by 5 %. The generalized purchasing power of undercompensated 51 840 mk/year under new prices p^1 would be 1.32 % less than the purchasing power of 50 000 mk/year under old prices p^0.

Note also that it would not be optimal for our consumer-saver to consume 80 % or 42 000 mk yearly and save the rest 10 500 mk out of annual income of 52 500 mk under p^1-prices, because in the truly compensated situation the optimal propensity to save was 18 %. Therefore the allocation (42 000,10 500) would produce somewhat less utility than the optimal allocation (43 050, 9 450) of the same compensated income $\tilde{Y}^1 = Y(p^1,V^0) =$ 52 500. Any allocation of some smaller yearly income (e.g. of 51 840 mk) would produce less utility.

Equation (118) shows that the price index of the disposable income $P(p^1,p^0;V^0)$ exceeds the consumer price index $\bar{P}(p^1,p^0;V^0)$ if the saving incentives in the new compensated situation under p^1-prices have diminished (i.e. $\bar{s}^1 < s^0$) as in our previous example. Intuitively, this means that the relative compensation of the disposable income must exceed the relative change in consumer prices $\bar{P}(p^1,p^0);V^0)$ if the subjective profitability of saving in the new price situation has diminished. Taking logarithms of both sides of (118) we get

$$(121) \qquad \log P(p^1,p^0;V^0) = \log \frac{Y(p^1,V^0)}{Y^0}$$

$$\approx \log\bar{P}(p^1,p^0;V^0) + \log(1 - a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0)$$

$$\approx \log\bar{P}(p^1,p^0;V^0) - a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0.$$

The relative change $\log\frac{Y(p^1,V^0}{Y^0}$ in the compensated income is thus divided into two terms, $\log\bar{P}(p^1,p^0;V^0)$ and $-a^0(\bar{s}^1 - s^0)(\bar{s}^1+s^0)Y^0$ The first term $\bar{P}(p^1,p^0;V^0)$ gives the standard (and main) effects of changing consumer prices. The latter term $-a^0(\bar{s}^1 - s^0)(\bar{s}^1 + s^0)Y^0$ is a correction factor which tells what is the relative change in disposable income needed to eliminate the changes in the subjective profitability of saving. It is positive if $\bar{s}^1 < s^0$. In our example $\log\bar{P}(p^1,p^0;V^0) = \log(1.046) = 0.045$ or 4.5 <u>%</u> (read 4.5 log-percents) and $-a^0(\bar{s}^1-s^0)(\bar{s}^1+s^0)Y^0 = 0.0038$ or 0.38 <u>%</u>. Therefore 0.38 <u>%</u> extra income is needed here to eliminate the reduction in the profitability of saving. Total change in the income must be thus 4.5 + 0.38 = 4.88 <u>%</u> corresponding to 5 % in ordinary percentages (as log(1.05) = 0.0488).

If prices are changed from p^0 to p^1 in such a way that saving incentives become stronger ($\bar{s}^1 > s^0$) then the correction term

$$(122) \qquad -a^0(\bar{s}^1 - s^0)(s^1 + s^0)Y^0$$

in (121) is negative and the needed change in the disposable income $\log \frac{Y(p^1,V^0)}{Y^0} = \log P(p^1,p^0;V^0)$ (i.e. the log-change in the price index of the disposable income) is smaller than the log-change in the consumer prices $\log \bar{P}(p^1,p^0;V^0)$.

Appendix 1. We will show here that if for all $(p,C) \in \mathbb{R}^{n+1}_{++}$ and $(\bar{p},\bar{C}) \in \mathbb{R}^{n+1}_{++}$ and for any $c \in \mathbb{R}_{++}$ the inequalities

$$(1) \qquad v(p,C|c) \geq v(\bar{p},\bar{C}|c)$$

$$(2) \qquad \bar{v}(p,C|c) \geq \bar{v}(\bar{p},\bar{C}|c)$$

hold simultaneously (i.e. either > or = in both), then there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ (depending possibly of c) such that for all $(p,C) \in \mathbb{R}^{n+1}_{++}$ and for any $c \in \mathbb{R}_{++}$

$$(3) \qquad v(p,C|c) = g(\bar{v}(p,C|c)) .$$

Equation (3) says that $v(p,C|c)$ is an increasing transformation of $\bar{v}(p,C|c)$ so that both functions are simultaneously ordinal indirect utility functions. By setting $c = 1$ we get the proposition following the inequalities (58).

Proof. Denote $z = (p,C)$ for convenience.
Let $A(v,c) = \{z|v(z|c) = v\}$ and $\bar{A}(\bar{v},c) = \{z|\bar{v}(z,c) = \bar{v}\}$ denote indifference surfaces of v and $\bar{v}$ in the space of $z = (p,C)$. If z and $\bar{z}$ are elements of $A(v,c)$ then $v(z|c) = v(\bar{z}|c)$ and because by assumption both (1) and (2) hold, also $\bar{v}(z|c) = \bar{v}(\bar{z}|c)$. Therefore $z, \bar{z} \in \bar{A}(\bar{v},c)$, where $\bar{v} = \bar{v}(z|c)$. This shows that the indifference surface $A(v(z|c),c)$ determined by any z is a subset of the other indifference surface $\bar{A}(\bar{v}(z|c),c)$ determined by the same z. It is shown similarly that $\bar{A}(\bar{v}(z|c),c) \subset A(v(z|c),c)$, so that for all possible v's

there exists a $\bar{v}$ such that $A(v,c) = \bar{A}(\bar{v},c)$. This means that the indifference surfaces of $v(z|c)$ and $\bar{v}(z|c)$ coincide or that the two functions remain constant in the same sets. Therefore there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $v(z|c) = g(\bar{v}(z|c))$, i.e. (3) holds. It remains to be shown that g is strictly increasing.

Suppose that it is not. Then we would have for some pair $(z,\bar{z})$

(4) $\quad \bar{v}(z|c) > \bar{v}(\bar{z}|c)$

(5) $\quad g(\bar{v}(z|c)) \leq g(\bar{v}(\bar{z}|c))$.

But the latter inequality implies

(6) $\quad v(z|c) \leq v(\bar{z}|c)$,

a contradiction with (1) and (2). Therefore g is necessarily also strictly increasing. $\square$

Appendix 2. Some regular utility functions of an income user. Consider any direct utility function $u(q)$ having properties B and its dual indirect utility

(1) $\quad v(p,C) = \max_q \{u(q) \mid p\cdot q \leq C\}$,

which will thus have properties $\bar{A}$. The demand function corresponding to these consumer preferencies is

(2) $\quad h(p,C) = h(r,1) = \frac{\nabla v(r,1)}{\nabla v(r,1)\cdot 1}$, $r = p/C$.

Consider first utility functions $\psi(p,Y,c)$ of an income user of the type

(3) $\quad \psi(p,Y,c) = v(p,cY) + \ell(c)$

or equivalently by (49)

(4) $\quad V(p,C,S) = \psi(p,C+S,\frac{C}{C+S})$

$$= v(p,C) + \ell(\frac{C}{C+S}) .$$

The preferences are regular because the function of (60)

(5) $\quad \bar{v}(p,C|c) = \psi(p,C/c,c)$

$$= v(p,C) + \ell(c)$$

has the properties of an indirect utility function for any $c > 0$. The demand function of the consumer for a given c is by (65b) and (2)

$$h(p,C|c) = h(r,1|c) \tag{6}$$

$$= \frac{\nabla\tilde{v}(r,1|c)}{\nabla\tilde{v}(r,1|c)\cdot r}$$

$$= \frac{\nabla v(r,1)}{\nabla v(r,1)\cdot 1}$$

$$= h(r,1)$$

$$= h(p,C)\ .$$

Thus it is independent of c and the consumption expenditure $C = cY$ is allocated into different consumer goods according to the arbitrary consumer demand function (2). We may say that the same consumer preferencies described by (1) - (2) apply uniformly for all c. This is a particularly beautiful special case of our theory, which contains the very generality of the standard consumer theory. In the role of a consumer the income user has an arbitrary demand function (2). Different increasing transformations

$$v^*(p,C) = g(v(p,C)) \tag{7}$$

of (1) lead to the same demand function (2) but the indifference curves

$$B(p,\psi) = \{(Y,c)\,|\,g(v(p,cY)) + \ell(c) = \psi\} \tag{8}$$

in the (Y,c)-space depend on the choice of $g(v)$ and $\ell(c)$. If we assume homotheticity (i.e. that $h(p,\lambda c) = \lambda h(p,C)$) and choose a special representation of $v(p,C)$ particularly simple results arise. Pick some reference prices $\tilde{p}$ and expenditure $\tilde{C}$ and represent $v(p,C)$ in the form

$$v^*(p,C) = g(v(p,C)) \tag{9}$$

$$= C/P(p)\ ,$$

where

$$P(p) = P(p,\tilde{p};\ \tilde{v}) = \frac{C(p,\tilde{v})}{C(\tilde{p},\tilde{v})} = \frac{C(p,\tilde{v})}{\tilde{C}} \tag{10}$$

is the Konüs cost of living index (3) comparing prices p with the reference prices $\tilde{p}$ on the reference utility level $\tilde{v} = v(\tilde{p},\tilde{C}) = u(h(\tilde{p},\tilde{C}))$. Here $C/P(p)$ is the real expenditure of the situation (p,C) and the representation (9) exists for any indirect utility function $v(p,C)$ representing homothetic consumer preferences. If e.g. in two situations (p^0,C^0) and (p^1,C^1) the real expenditures are equal, $C^0/P(p^0) = C^1/P(p^1)$, then

$$C^1/C^0 = P(p^1)/P(p^0) \tag{11}$$

$$= \frac{C(p^1,\tilde{v})}{\tilde{C}} \cdot \frac{\tilde{C}}{C(p^0,\tilde{v})}$$

$$= P(p^1,p^0;\ \tilde{v})\ ,$$

where $P(p^1,p^0;\tilde{v})$ is independent of the reference utility level $\tilde{v}$ because of homotheticity. This states that the change $C^0 \to C^1$ in expenditure is just enough to compensate for the price change $p^0 \to p^1$ or that the utility has remained the same, $v(p^0,C^0) = v(p^1,C^1)$, see e.g. Vartia (1978b). Now let

(12) $\psi(p,Y,c) = c\frac{Y}{P(p)} + \ell(c)$

with indifference curves

(13) $B(p,\psi) = \{(Y,c) \mid c\frac{Y}{P(p)} + \ell(c) = \psi\}$.

The indifference curve $B(p,\psi)$ has a simple representation

(14) $Y = P(p)(\frac{\psi - \ell(c)}{c})$.

From (14) we may easily calculate all pairs (Y,c) of $B(p,\psi)$. But we may also infer how $B(p^0,\psi)$ and $B(p^1,\psi)$ are related. Let $(Y^0,c) \in B(p^0,\psi)$ and $(Y^1,c) \in B(p^1,\psi)$. From (14) we see a remarkable fact that their ratio is independent of both c and ψ:

(15) $Y^1/Y^0 = P(p^1)/P(p^0)$.

It depends only on prices p^0 and p^1. Therefore for preferences of the type (12) two indifference curves $B(p^0,\psi)$ and $B(p^1,\psi)$ corresponding to the same utility ψ are related in an extremely simple way:

(16) $(Y^0,c) \in B(p^0,\psi)$ & $(Y^1,c) \in B(p^1,\psi) \Rightarrow Y^1 = \lambda Y^0$,

where $\lambda = P(p^1)/P(p^0)$. The same stretching factor λ applies for all c's and for all utility levels ψ. We may call $y = Y/P(p)$ the real income and investigate the indifference curves on the space of $(y,c) = (\frac{Y}{P(p)}, c)$. Because by (13)

(17) $y = \frac{Y}{P(p)} = \frac{\psi - \ell(c)}{c}$

the indifference curves on (y,c)-space are

(18) $B_R(p,\psi) = \{(y,c) \mid y = \frac{\psi - \ell(c)}{c}\}$

The images of $B(p^0,\psi)$ and $B(p^1,\psi)$ coincide on the real income scale,

(19) $B_R(p^0,\psi) = B_R(p^1,\psi)$,

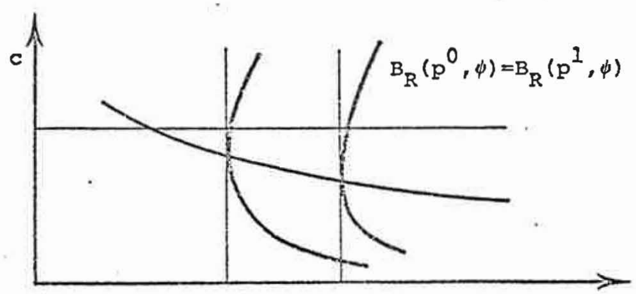
which shows that $B_R(p,\psi)$ of (12) is independent of p! We will call a utility function $\psi(p,Y,c)$ and the implied preferences strongly p-regular if (19) holds, or equivalently: for all p^0 and p^1 there exists a $\lambda > 0$ such that for all $Y^0 > 0$ and $c > 0$

(20) $\psi(p^0,Y^0,c) = \psi(p^1,\lambda Y^0,c)$.

Of course, strong p-regularity does not hold generally. Note also that for any regular or p-regular $\psi(p,Y,c)$ equation (16) holds for proportional changes in prices, $p^0 \to \lambda p^0$, and therefore

(21) $$B_R(p^0,\psi) = B_R(\lambda p^0,\psi) .$$

Figure 13. Indifference curves $B_R(p,\psi)$ of some strongly p-regular preferences



For preferences which are not strongly p-regular $B_R(p^0,\psi)$ and $B_R(p^1,\psi)$ do not coincide complitely although they normally situate near each other.

We may generate different strongly p-regular preferences all of which lead to the same homothetic consumer demand $h(p,C)$ by choosing $\ell(c)$ in (12) in a suitable way. The indifference curves (18) may be made to change in a very general way when utility ψ or equivalently real income $y = Y/P(p)$ changes. Optimal c is the solution of

(22) $$\partial\psi(p,Y,c)/\partial c = \psi c = Y/P(p) + \ell'(c) = 0 ,$$

from which we see that $c = \hat{c}$ depends only on real income $y = Y/P(p)$. If $-\ell'(c) = \alpha+\beta c$ we have

(23) $$y = \alpha + \beta c = L(c) ,$$

i.e., real income y is a linear function $L(\hat{c}) = \alpha + \beta\hat{c}$ of the optimal $\hat{c}$. Inversely, $\hat{c}$ as a function of y is also linear

(24) $$\hat{c} = H(y) = \frac{1}{\beta}(y-\alpha) = \frac{1}{\beta}(\frac{Y}{P(p)} - \alpha) .$$

By integrating

(25) $$\ell'(c) = -\alpha - \beta c$$

we get

(26) $$\ell(c) = d - \alpha c - \frac{\beta}{2}c^2 .$$

By normalization $\ell(1) = 0$ we have $d = \alpha + \frac{\beta}{2}$ and finally

(27) $$\psi(p,Y,c) = c\frac{Y}{P(p)} + \alpha(1-c) + \frac{\beta}{2}(1-c^2).$$

For preferences (27) optimal $\hat{c}$ is a linear function of real income $y = Y/P(p)$ and $\hat{c} = 1$ when $y = \alpha$. If $\beta < 0$ the optimal propensity to consume diminishes with real income. For any (p,Y,c) where c may or may not be optimal the demand function is

(28) $h(p,C) = h(r,1) = \dfrac{V\frac{1}{P(r)}}{V\frac{1}{P(r)} \cdot r}$,

where $r = p/C$ and $C = cY$. It is necessarily homothetic: $h(p,\psi C) = \psi h(p,C)$ for all admissible arguments.

As the second example we specify

(29) $y = -\ell'(c) = L(c) = \dfrac{c-a}{dc-b}$

which corresponds to quite a flexible function

(30) $\hat{c} = H(y) = \dfrac{a+by}{1+ay}$.

Integrating (29) gives

(31) $\ell(c) = e + \int(\dfrac{a-c}{b-dc})dc$

$= e - (\frac{a}{d}+b)\log|b-dc| - \frac{c}{d}$.

If we generalize (12) to

(32) $\psi(p,Y,c) = f(c)\dfrac{cY}{P(p)} + \ell(c)$

we maintain strong p-regularity as $(y,c) \in B_R(p,\psi)$ implies

(33) $y = (\psi - \ell(c))/f(c)c$.

If on the other hand

(34) $\psi(p,Y,c) = g(\dfrac{cY}{P(p)}] + \ell(c)$

the indifference curve on (y,c)-scale satisfies

(35) $g(cy) = \psi - \ell(c)$.

Because g(v) is increasing we may solve for y

(36) $y = g^{-1}(\psi - \ell(c))/c$

so that also here $B_R(p,\psi)$ is independent of p and strong p-regularity holds. More generally, $\psi(p,Y,c)$ is strongly p-regular if

(37) $\psi(p,Y,c) = f(v(p,cY),c)$,

where v(p,C) represents <u>homothetic</u> consumer preferences and is thus of the form $v(p,C) = g(C/P(p))$, see Afriat (1972, p. 26).

Let's generalize (34) for nonhomothetic preferences:

(38) $\psi(p,Y,c) = g(v(p,cY)) + \ell(c)$.

The indifference curve in (Y,c)-space is

(39) $B(p,\psi) = \{(Y,c)|v(p,cY) = g^{-1}(\psi - \ell(c))\}$

and it doesn't have the property (16) unless $v(p,C)$ is homothetic. If $(Y^0,c) \in B(p^0,\psi)$ and $(Y^1,c) \in B(p^1,\psi)$ we have by (39)

$$(40) \qquad g^{-1}(\psi - \ell(c)) = v(p^0,cY^0) = v(p^1,cY^1) .$$

But by the definition of the Konüs cost of living index corresponding to consumer preferencies $v(p,C)$ we have

$$(41) \qquad \frac{cY^1}{cY^0} = \frac{Y^1}{Y^0} = P(p^1,p^0;\ g^{-1}(\psi - \ell(c)) .$$

Unless $v(p,C)$ is homothetic the ratio Y^1/Y^0 depends not only of p^1 and p^1 but also of the utility level $\tilde{v} = g^{-1}(\psi - \ell(c))$. But usually this dependance is numerically small so that (16) holds approximately. Therefore if some $B(p^0,\psi)$ is specified from (39) after choosing some functions $g(v)$ and $\ell(c)$ for a given $v(p,cY)$ all $B(p,\psi)$'s with $p \approx \lambda p^0$ are of a similar shape. Optimal $\hat{c}$ is the solution of

$$(42) \qquad g'(v(p,cY))\frac{\partial v(p,cY)}{\partial cY} = -\ell'(c)/c .$$

By suitable choices of $v(p,cY)$, $g(v)$ and $\ell(c)$ very general kind of dependance between $\hat{c}$ and Y for a given p may be generated. For any $C = cY$ the consumer demand function satisfies (6) so that any (nonhomothetic or homothetic) consumer behaviour may be realized by spesification (38).

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